(a)

$$3x^2+1=-5x$$

(by completing the square)

$$3x^2+5x+1=0$$

$$x^2+\frac{5}{3}x+\frac{1}{3}=0$$

$$\left[x^2+\frac{5}{3}x+\left(\frac{5}{6}\right)^2\right]+\frac{1}{3}-\left(\frac{5}{6}\right)^2=0$$

$$\left(x+\frac{5}{6}\right)^2=\frac{25}{36}-\frac{12}{36}$$

$$\left(x+\frac{5}{6}\right)^2=\frac{13}{36}$$

$$x+\frac{5}{6}=\pm\frac{\sqrt{13}}{6}$$

$$\boxed{x=\frac{-5\pm\sqrt{13}}{6}}$$

(b)

$$x^4-2=4x^2$$

$$x^4-4x^2-2=0$$

$$(x^2)^2-4(x^2)-2=0$$

Let $u=x^2$ } note, u must be positive

$$u^2-4u-2=0$$

$$u=\frac{4\pm\sqrt{16+8}}{2}=\frac{4\pm\sqrt{24}}{2}$$

$$=\frac{4\pm2\sqrt{6}}{2}=2\pm\sqrt{6}$$

But $u=2-\sqrt{6}$ is negative, so it can't equal any x^2.

So $x^2=2+\sqrt{6}$

$$\boxed{x=\pm\sqrt{2+\sqrt{6}}}$$

(c)

$$\log_2(x+5)+\log_2(x-3)=\log_2 3$$

Don't forget to check answers for domain issues!

$$\log_2(x+5)(x-3)=\log_2 3$$

$$\cancel{2}^{\cancel{\log_2}(x+5)(x-3)}=\cancel{2}^{\cancel{\log_2}3}$$

$$(x+5)(x-3)=3$$

$$x^2+2x-15=3$$

$$x^2+2x-18=0$$

Let's use Po-Shen Loh..

$$r_1+r_2=-2 \rightarrow r_{avg}=\frac{-2}{2}=-1$$

$$r_1\cdot r_2=-18$$

$$\begin{cases} r_1=(-1+u) \\ r_2=(-1-u) \end{cases}$$

$$(-1+u)(-1-u)=-18$$

$$1-u^2=-18$$

$$u^2=19$$

$$u=\pm\sqrt{19} \rightarrow x=-1\pm\sqrt{19}$$

But $-1-\sqrt{19}$ causes a domain issue in 2^{nd} log of original equation

So $\boxed{x=-1+\sqrt{19}}$

(d)

$$x^2-4x=1$$

(use Po Shen Loh)

$$x^2-4x-1=0$$

$$r_1+r_2=4 \rightarrow r_{avg}=\frac{4}{2}=2$$

$$r_1\cdot r_2=-1$$

$$\begin{cases} r_1=2+u \\ r_2=2-u \end{cases}$$

$$(2+u)(2-u)=-1$$

$$4-u^2=-1$$

$$u^2=5$$

$$u=\pm\sqrt{5}$$

$$r_1=2+\sqrt{5}, \quad r_2=2-\sqrt{5}$$

$$\boxed{x=2\pm\sqrt{5}}$$

(e)

$$5t^2-8t=3$$

(by completing the square)

$$5\left(t^2-\frac{8}{5}t+\frac{16}{25}\right)-\frac{16}{5}=3$$

$$5\left(t-\frac{4}{5}\right)^2-\frac{16}{5}=3$$

← Socks and Shoes from here!

$$5\left(t-\frac{4}{5}\right)^2=\frac{31}{5}$$

$$\left(t-\frac{4}{5}\right)^2=\frac{31}{25}$$

$$t-\frac{4}{5}=\pm\frac{\sqrt{31}}{5} \rightarrow \boxed{t=\frac{4\pm\sqrt{31}}{5}}$$

(f)

$$\frac{x}{x+3}+\frac{x}{x+5}=\frac{2}{x^2+8x+15}$$

Be aware of domain.. denom must not be zero!

$$\frac{x}{x+3}+\frac{x}{x+5}=\frac{2}{(x+3)(x+5)}$$

$$\frac{x}{(x+3)}\cdot\frac{(x+5)}{(x+5)}+\frac{x}{(x+5)}\cdot\frac{(x+3)}{(x+3)}=\frac{2}{(x+3)(x+5)}$$

$$x(x+5)+x(x+3)=2$$

$$x^2+5x+x^2+3x=2$$

$$2x^2+8x-2=0$$

Let's use Po-Shen Loh for fun...

$$x^2+4x-1=0$$

← divide by 2 to get leading coef. = 1

$$r_1+r_2=-4 \rightarrow r_{avg}=\frac{-4}{2}=-2$$

$$r_1\cdot r_2=-1$$

$$\begin{cases} r_1=(-2+u) \\ r_2=(-2-u) \end{cases}$$

$$(-2+u)(-2-u)=-1$$

$$4-u^2=-1$$

$$u^2=5$$

$$u=\pm\sqrt{5}$$

$$\boxed{x=-2\pm\sqrt{5}}$$

thankfully, no domain issues

(g) $5xe^{2x}=3$

(using Lambert's W)

we want to see $u\cdot e^u$ for some u. We almost have $(2x)\cdot e^{(2x)}$, except the first 2 is a 5 in our equation. so let's insert a 2 and compensate for our insertion...

$$5\cdot\frac{2xe^{2x}}{2}=3$$

$$5\cdot\frac{W^{-1}(2x)}{2}=3$$

Socks and Shoes from here!

$$\frac{W^{-1}(2x)}{2}=\frac{3}{5}$$

$$W^{-1}(2x)=\frac{6}{5}$$

$$2x=W\left(\frac{6}{5}\right)$$

$$\boxed{x=\frac{W\left(\frac{6}{5}\right)}{2}}$$

Note as $\frac{6}{5}>0$ we only need to worry about a solution on the 1st "branch" (i.e, we don't need a separate case where W_{-1} is applied to both sides)

(h) $e^{2x}-7e^x+5=0$

note $e^{2x}=(e^x)^2$ so let's use a u-substitution of $u=e^x$

$$(e^x)^2-7e^x+5=0$$

$$u^2-7u+5=0$$

we try to factor the above, but to no avail. Hence, we should try either "completing the square" or Po-shen Loh. Let's do the latter...

$r_1+r_2=7\rightarrow r_{avg}=\frac{7}{2}$

$r_1\cdot r_2=5$

$r_1=\frac{7}{2}-u$

$r_2=\frac{7}{2}+u$

$$\left(\frac{7}{2}-u\right)\left(\frac{7}{2}+u\right)=5$$

$$\frac{49}{4}-u^2=5$$ ← Sock and shoes from here!

$$-u^2=5-\frac{49}{4}$$

$$-u^2=\frac{-29}{4}$$

$$u^2=\frac{29}{4}$$

$$u=\frac{\pm\sqrt{29}}{2}$$

$$\boxed{x=\frac{7\pm\sqrt{29}}{2}}$$

(i)

$$\log_3(2x+1)-\log_3(x-4)=\log_3 x$$

Be aware, given the potential domain issues with the $\log_3$ function we should check any potential solutions found

Using the zero-product property looks difficult, as the logarithms don't lend themselves to easy factoring. So let's try to work towards "sock and shoes" by getting all the x variables together where they can better interact - then maybe we'll see something...

$$\log_3(2x+1)-\log_3(x-4)-\log_3 x=0$$

$$\log_3\left[\frac{2x+1}{x(x-4)}\right]=0$$

Recall the log properties:

$\log_3 A+\log_3 B=\log_3 AB$

$\log_3 A-\log_3 B=\log_3\frac{A}{B}$

$\log_3 A^p=p\log_3 A$

now kill off the $\log_3$ with exponentiation, base 3 on both sides to make this simpler!

$$\cancel{3^{\log_3}}\left[\frac{2x+1}{x(x-4)}\right]=3^0$$

$$\frac{2x+1}{x(x-4)}=1$$

This should be solvable by zero-product property since we have many tools to factor the polynomials in a rational expression - just don't forget to get zero on one side first!

$$\frac{2x+1}{x(x-4)}-1=0$$

$$\frac{2x+1}{x(x-4)}-\frac{x(x-4)}{x(x-4)}=0$$

$$\frac{2x+1-x(x-4)}{x(x-4)}=0$$

$$\frac{-x^2+6x+1}{x(x-4)}=0$$

$-x^2+6x+1=0$ and $x\neq 0,4$

lest we divide by zero here

darn! the left side doesn't factor nicely! No matter. it's quadratic - so let's "complete the square"

(alternatively, we could also use Po-Shen Loh)

$$x^2-6x-1=0$$

$$(x-6x+9)-9-1=0$$

$$(x-3)^2-10=0$$ Finally! Socks and Shoes!

$$(x-3)^2=10$$

$$x-3=\pm\sqrt{10}$$

$$x=3\pm\sqrt{10}$$

But don't forget

Check if there were any domain issues! $\log_3 x$ is undefined if $x<0$

So, $\boxed{x=3+\sqrt{10}}$

(j) $\sqrt{\log_{\frac{1}{2}}(x)-3}+\sqrt{\log_{\frac{1}{2}}(x)}=3$

Seeing $\log_{\frac{1}{2}}(x)$ in two places makes us wonder if u-substitution might help reveal something...

Let $u=\log_{\frac{1}{2}}(x)$. Then

$\sqrt{u-3}+\sqrt{u}=3$

Using the zero product property seems difficult, but maybe we could reduce the number of u variables by "freeing" them from their square root "cages" so they can better interact.

Remember – squaring both sides as they currently stand would produce an undesirable middle term of $2\sqrt{u^2-3u}$. So isolate each radical in turn and only then, once you have one isolated, should you square both sides...

$\sqrt{u-3}=3-\sqrt{u}$

$(\sqrt{u-3})^2=(3-\sqrt{u})^2$

$\cancel{u}-3=9-6\sqrt{u}+\cancel{u}$

Note: domain issues with the square root and the non-invertibility of squaring will require checking the solutions this produces!

Starting to isolate the $\sqrt{u}$ by moving the u on the right to the left (by subtracting u from both sides) leaves an equation with only one occurrence of u. Socks and Shoes time!

$-12=-6\sqrt{u}$

$2=\sqrt{u}$

$u=4$

Don't forget to check!

$\sqrt{u-3}\stackrel{?}{=}3-\sqrt{u}$

$\sqrt{4-3}=1$ | $3-\sqrt{4}=1$ ✓

We're good.

Don't forget – we want a solution for x, not u

$4=\log_{\frac{1}{2}}(x)$

This step is not essential, but remembering the 2 properties below can sometimes make the numbers nicer

1) $\log_b x^p=p\log_b x$

2) $\log_{b^p} x=\frac{1}{p}\log_b x$

$4=-\log_2 x$

Socks and Shoes!

$-4=\log_2 x$

$2^{-4}=\cancel{2^{\log_2}}x$

$x=2^{-4}$

$\boxed{x=\frac{1}{16}}$

don't forget to check to ensure no domain issues! but we are fine here

Check: $\sqrt{\log_{\frac{1}{2}}\left(\frac{1}{16}\right)-3}+\sqrt{\log_{\frac{1}{2}}\left(\frac{1}{16}\right)}$

$=\sqrt{4-3}+\sqrt{4}$

$=1+2$

$=3$ ✓

We're Good! ☺

(k) $2x+\ln x=3$

(using Lambert's W)

exponentiate both sides, base e, to introduce e to a power

$e^{2x+\ln x}=3$

$e^{2x}\cdot e^{\ln x}=3$

$e^{\ln x}$ reduces to x which might help — so break this apart and simplify

$e^{2x}\cdot x=3$

$2x\cdot e^{2x}=6$

We are missing just a 2 to manufacture $W^{-1}(u)=u\cdot e^u$ for some u (here $u=2x$)

$W^{-1}(2x)=6$

$2x=W(6)$

noting $6>0$ we can confine ourselves to the first branch.

$\boxed{x=\frac{W(6)}{2}}$