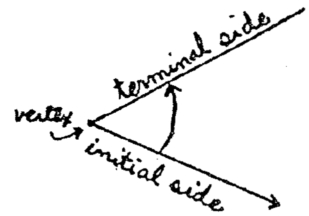
NOTES ON TRANSCENDENTAL FUNCTIONS

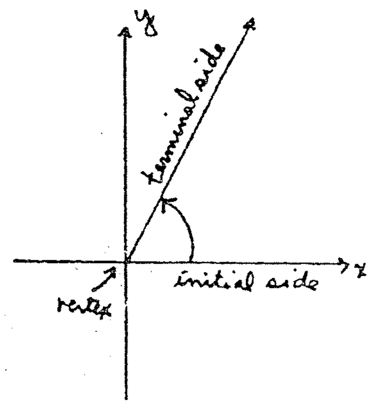
Section A. Introduction to Trigonometry

I. The Measurement of Angles

An *angle* is formed by two half-lines having a common endpoint called the *vertex*. One half-line is designated the *initial side* and the other the *terminal side*. We may consider the angle as having been formed by a rotation from the initial to the terminal side.



When we place the angle in the Cartesian coordinate plane with its vertex at the origin and initial side coinciding with the positive x-axis, we say that the angle is in *standard position*. In future discussions we will consider angles in their standard positions.



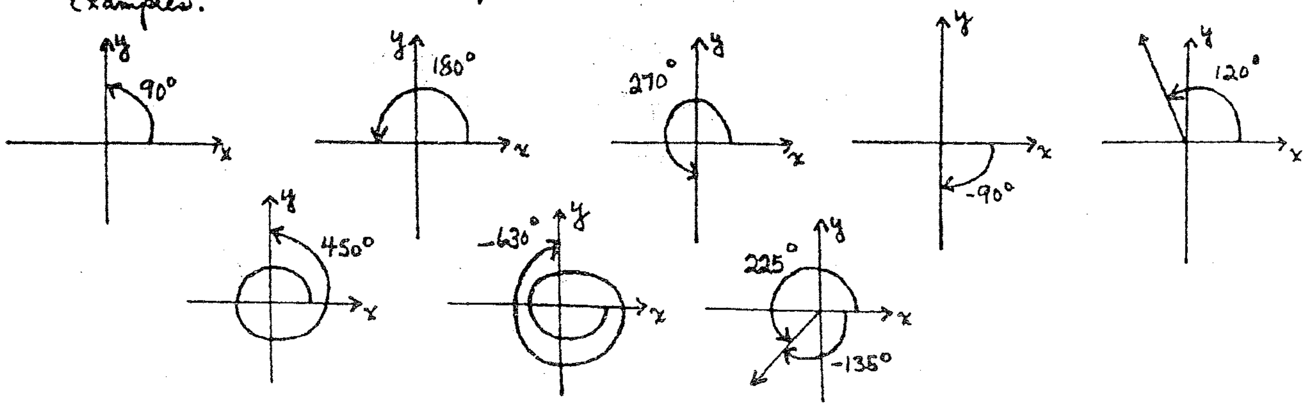
Angle in Standard Position

a. Degree Measure of Angles

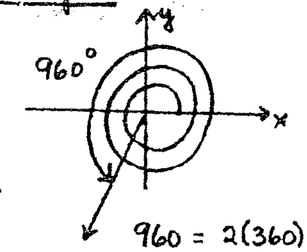
The angle formed by one complete *counterclockwise* rotation is assigned degree measure 360°. Other angles are then assigned degree measure based on this. The sign of the measure is

+ if rotation is counterclockwise
− if rotation is clockwise

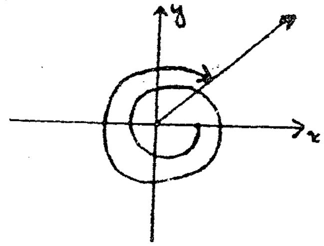
Examples:



Example 1. Locate the terminal side of angles of $960°$ and $-675°$



$960 = 2(360) + 240$



$675 = 1(360) + 315$

b. Coterminal Angles

When two angles have the same initial and terminal sides, they are said to be <u>Coterminal</u> angles. From Example 1, we see that

angles of $960°$ and $240°$ are coterminal angles

angles of $-675°$, $-315°$, $45°$ are coterminal angles

In general, the angle θ and the angle $\theta \pm n(360°)$, for $n = 1, 2, 3, \cdots$ are coterminal.

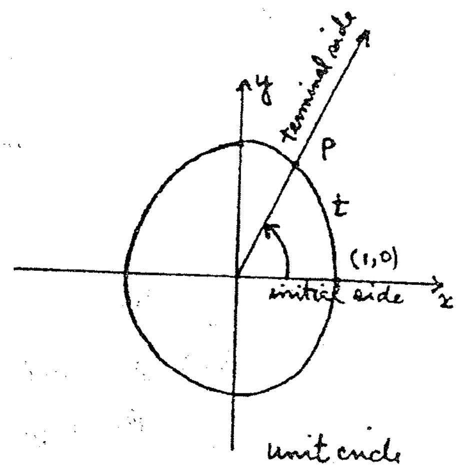
c. Radian Measure of Angles

The radian measure of an angle is based on the length of an arc on the <u>unit</u> circle

$$x^2+y^2=1.$$

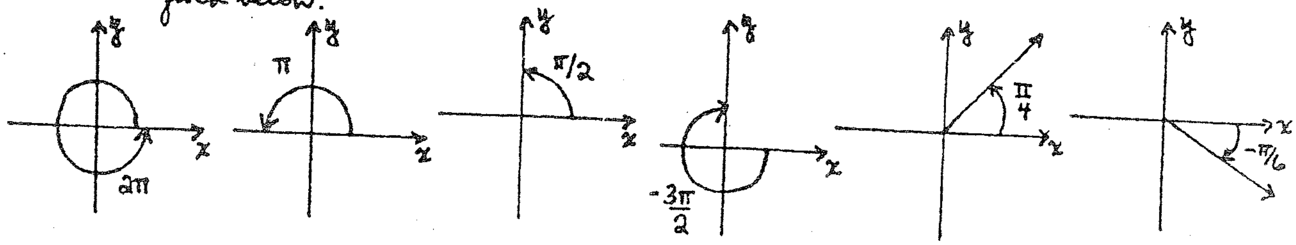
We measure the angle by the distance t traversed along the circumference of the unit circle as the angle is generated by a rotation from the initial to the terminal side in standard position. The sign conventions are as before

+ counterclockwise
- clockwise



unit circle
$x^2+y^2=1$

The radian measure of the angle formed by one complete counterclockwise rotation equals the circumference of the unit circle $= 2\pi(1) = 2\pi$. Other examples are given below.



d. Conversion Formula : $\boxed{\pi \text{ radians} = 180^\circ}$

$$\text{so } 1^\circ = \frac{\pi}{180} \text{ radians}$$

$$1 \text{ radian} = \frac{180}{\pi}^\circ$$

Example 2. Convert to radians: 20°, 30°, -60°

$$20^\circ = 20\left(\frac{\pi}{180}\right) \text{radians} = \frac{\pi}{9} \text{ radians}$$

$$30^\circ = 30\left(\frac{\pi}{180}\right) \text{radians} = \frac{\pi}{6} \text{ radians}$$

$$-60^\circ = -60\left(\frac{\pi}{180}\right) \text{radians} = -\frac{\pi}{3} \text{ radians}$$

Example 3. Convert to degrees: $\frac{7\pi}{6}$ radians, $-\frac{\pi}{12}$ radians, 0.76 radians

$$\frac{7\pi}{6} \text{ radians} = \frac{7\pi}{6} \cdot \left(\frac{180}{\pi}\right)^\circ = 210^\circ$$

$$\frac{-\pi}{12} \text{ radians} = \frac{-\pi}{12}\left(\frac{180}{\pi}\right)^\circ = -15^\circ$$

$$0.76 \text{ radians} = 0.76\left(\frac{180}{\pi}\right)^\circ = \left(\frac{136.8}{\pi}\right)^\circ \approx 43.54^\circ$$

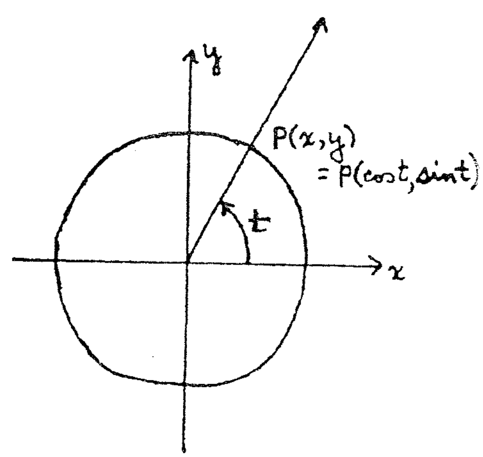
II. The Sine and Cosine Functions

a. Definitions

Given a real number t, there corresponds an angle of t radians. Let $P(x,y)$ be the point of intersection of the terminal side of the angle t (in standard position) with the unit circle. Then the cosine and sine functions of the number t are defined as follows:

$$\cos t = x$$

$$\sin t = y$$



unit circle

$x^2+y^2=1$

b. Some Basic Properties of the Sine and Cosine Functions

(1) The <u>domains</u> of both the sine and cosine functions consist of all real numbers.

(2) The <u>ranges</u> of both the sine and cosine functions consist of all real numbers between -1 and 1, inclusive.

$$-1 \le \sin t \le 1 \quad \text{and} \quad -1 \le \cos t \le 1 \quad \text{for all } t$$

(3) Fundamental Identity

$$\cos^2 t + \sin^2 t = 1 \quad \text{for all } t$$

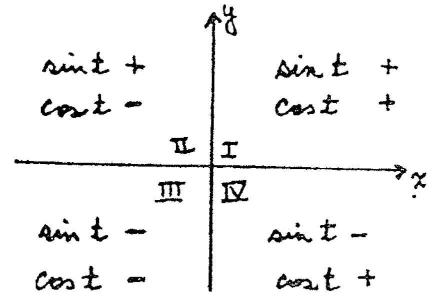
[since $x^2+y^2=1$]

(4) Periodicity

$$\left.\begin{aligned} \sin(t \pm 2n\pi) &= \sin t \\ \cos(t \pm 2n\pi) &= \cos t \end{aligned}\right\} \text{ where } n = 1, 2, 3, \ldots$$

[coterminal angles have same sine and cosine function values]

(5) <u>Signs</u> of the sine and cosine functions for t an angle with terminal side in each of the four quadrants are shown in the figure on the right.



<u>Example 4</u>. Given that $\sin t = \frac{1}{4}$, and that the terminal side of the angle of t radians is in the 2^{nd} quadrant, find $\cos t$.

$\cos^2 t + \sin^2 t = 1$ (Fund. Identity)

$\sin t = \frac{1}{4}$ (Given)

Then $\cos^2 t + \left(\frac{1}{4}\right)^2 = 1$

$\cos^2 t = 1 - \frac{1}{16} = \frac{15}{16} \rightarrow \cos t = \pm\sqrt{\frac{15}{16}} = \pm\frac{\sqrt{15}}{4}$ } but $\cos t$ is negative in quad. II. Hence $\boxed{\cos t = -\frac{\sqrt{15}}{4}}$

<u>Example 5</u>. If $\cos t = \frac{3}{10}$, find all possible values of $\sin t$.

6. Values of the Sine and Cosine Functions [remember $\cos t = x$; $\sin t = y$]

(1) Integer multiples of $\frac{\pi}{2}$

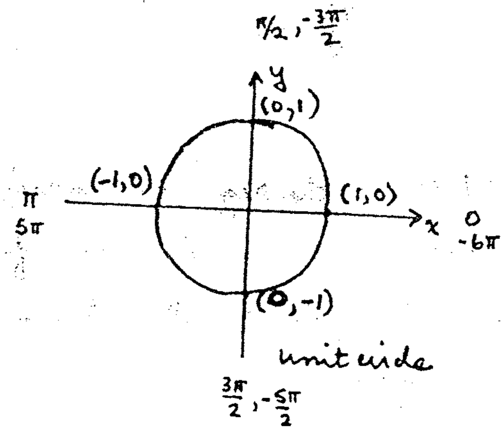
From the figure on the right we have

$\cos 0 = 1$ and $\sin 0 = 0$

$\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$

$\cos \pi = -1$ and $\sin \pi = 0$

$\cos \frac{3\pi}{2} = 0$ and $\sin \frac{3\pi}{2} = -1$



Also, for example,

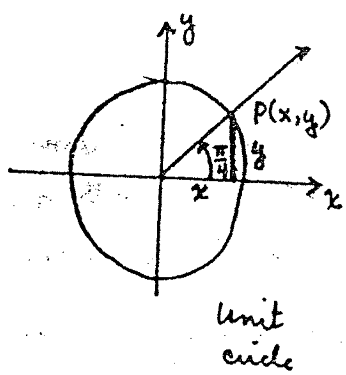
$\cos(-\frac{3\pi}{2}) = 1$, $\sin(5\pi) = 0$, $\cos(-6\pi) = 1$, $\sin(-\frac{5\pi}{2}) = -1$

(2) Important first-quadrant angles

<u>$t = \frac{\pi}{4}$</u> (45°)

$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$



$x^2 + y^2 = 1$

but $x = y$ (45° angle)

so $x^2 + x^2 = 1 \rightarrow 2x^2 = 1 \rightarrow x^2 = \frac{1}{2}$

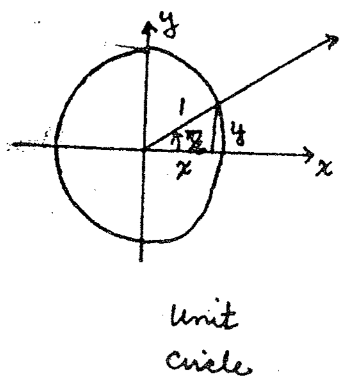
so $x = \pm\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$y = x = \frac{\sqrt{2}}{2}$

<u>$t = \frac{\pi}{6}$</u> (30°)

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sin \frac{\pi}{6} = \frac{1}{2}$



$x^2 + y^2 = 1$

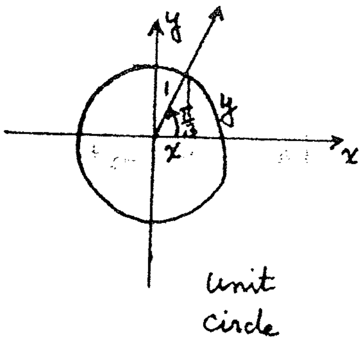
but $y = \frac{1}{2}$ (side opposite the 30° angle is $\frac{1}{2}$ the hypotenuse in a 30°-60° right Δ)

so $x^2 + (\frac{1}{2})^2 = 1$

$x^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$x = \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$; $y = \frac{1}{2}$

$t = \frac{\pi}{3}$ (60°)



$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$x^2+y^2=1$

but $x = \frac{1}{2}$ (this time x is the side opposite at 30° angle)

so $y^2 = \frac{3}{4} \rightarrow y = {}^{+}\frac{\sqrt{3}}{2}$; $x = \frac{1}{2}$

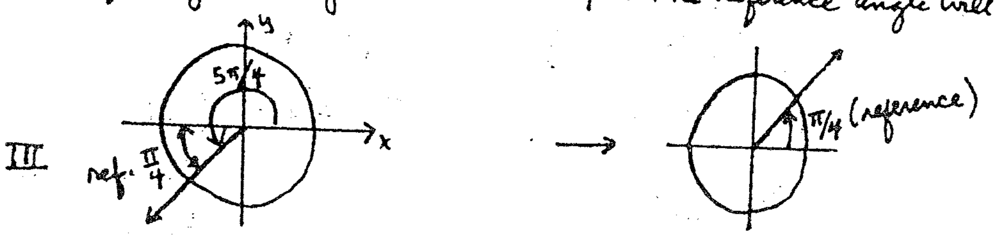
The table on the right summarizes the values of sine and cosine for the first quadrant. It should be memorized.

t	cos t	sin t
0	1	0
$\pi/6$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\pi/4$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\pi/3$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\pi/2$	0	1

(3) Values of Sine and Cosine for Other Important Numbers (integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$)

To find the values of the sine and cosine functions of angles whose terminal sides lie in the 2nd, 3rd, and 4th quadrants we relate the given angle to a first-quadrant angle, as follows ($\frac{5\pi}{4}$ used as example)

- First, construct the angle in question, and note quadrant.
- Second, find the associated reference angle, the angle between the terminal side of the given angle and the X-axis. The reference angle will be acute.



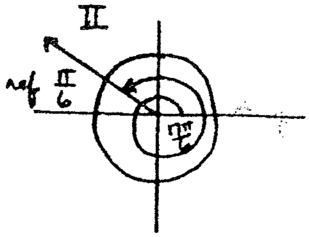
- Third find the sine and/or cosine of the reference angle [have the same absolute value as the functions of the given angle]

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\pi/4 = \sqrt{2}/2$$

- Fourth, adjust the signs according the the quadrant of the given angle.

$$\left.\begin{aligned} \cos\frac{5\pi}{4} &= -\sqrt{2}/2 \\ \sin\frac{5\pi}{4} &= -\sqrt{2}/2 \end{aligned}\right\}$$ in 3rd quad. both are −

Example 6. Find the value of $\sin \frac{17\pi}{6}$



ref. angle $= \frac{\pi}{6}$

$\sin \frac{\pi}{6} = \frac{1}{2}$, so $\sin \frac{17\pi}{6} = \frac{1}{2}$ (+ because sine is positive in 2^{nd} quadrant)

Example 7. Find the value of $\cos(-\pi/3)$

Example 8. Find the value of $\sin\left(-\frac{5\pi}{2}\right)$

III. The Other Trigonometric Functions

a. Definitions

The tangent, secant, cotangent, and cosecant functions are defined in terms of the sine and cosine functions, as follows:

$$\left.\begin{aligned} \tan t &= \frac{\sin t}{\cos t} \\ \sec t &= \frac{1}{\cos t} \end{aligned}\right\} \text{for all } t \text{ except } \frac{\pi}{2} \pm n\pi,\ n = 0, 1, 2, \dots \ \left(\text{odd multiples of } \frac{\pi}{2}\right)$$

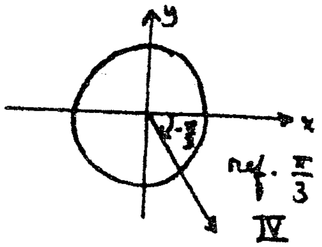
$$\left.\begin{aligned} \cot t &= \frac{\cos t}{\sin t} \\ \csc t &= \frac{1}{\sin t} \end{aligned}\right\} \text{for all } t \text{ except } \pm n\pi,\ n = 0, 1, 2, \dots \ (\text{integer multiples of } \pi)$$

Note the restrictions on the domains of these other trigonometric functions. The values of t excluded from their domains are those for which the denominators in their definitions are equal to zero.

b. Values of the other trigonometric functions

Since the other trigonometric functions are defined in terms of the sine and cosine, values of these other functions can easily be obtained for any t for which the sine and cosine are known.

Example 9. Find $\tan t$, $\cot t$, $\sec t$, and $\csc t$ for $t = -\frac{\pi}{3}$.



reference angle $= \frac{\pi}{3}$

$$\left.\begin{aligned} \cos\frac{\pi}{3} &= \frac{1}{2} \\ \sin\frac{\pi}{3} &= \frac{\sqrt{3}}{2} \end{aligned}\right\} \longrightarrow \left.\begin{aligned} \cos\left(-\frac{\pi}{3}\right) &= \frac{1}{2} \\ \sin\left(-\frac{\pi}{3}\right) &= -\frac{\sqrt{3}}{2} \end{aligned}\right\} \text{after IV-quad. adjustment of signs}$$

Then

$$\tan\left(-\frac{\pi}{3}\right) = \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos(-\pi/3)} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

$$\cot(-\pi/3) = \frac{\cos(-\pi/3)}{\sin(-\pi/3)} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec(-\pi/3) = \frac{1}{\cos\left(-\frac{\pi}{3}\right)} = \frac{1}{1/2} = 2$$

$$\csc(-\pi/3) = \frac{1}{\sin\left(-\frac{\pi}{3}\right)} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

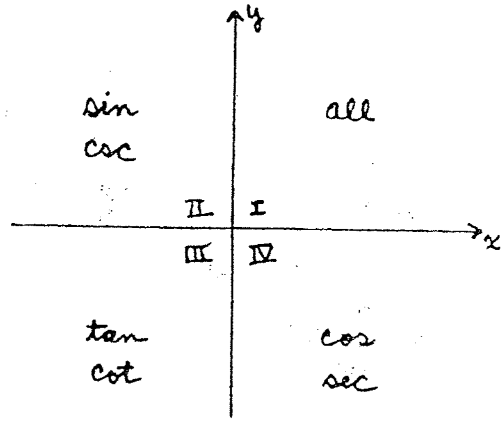
Example 10. Find $\sec\frac{7\pi}{6}$.

Example 11. Find $\csc\left(-\frac{3\pi}{2}\right)$ and $\cot\left(-\frac{3\pi}{2}\right)$.

Example 12. Find $\tan\left(\frac{3\pi}{4}\right)$.

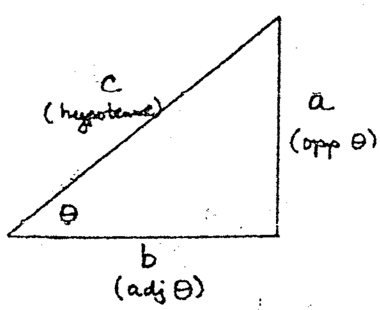
c. Signs of the trigonometric functions in the four quadrants (a summary).

The figure on the right shows, for each quadrant, which functions are <u>positive</u> when t is an angle with terminal side in that quadrant.



IV. Right-Triangle Trigonometry

In a right triangle, the following relations hold:



$$a^2 + b^2 = c^2$$

$$\sin\theta = \frac{a}{c} = \frac{\text{opp}}{\text{hyp}}$$

$$\cos\theta = \frac{b}{c} = \frac{\text{adj}}{\text{hyp}}$$

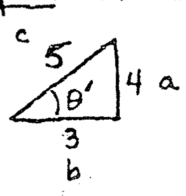
$$\tan\theta = \frac{a}{b} = \frac{\text{opp}}{\text{adj}}$$

$$\cot\theta = \frac{b}{a} = \frac{\text{adj}}{\text{opp}}$$

$$\sec\theta = \frac{c}{b} = \frac{\text{hyp}}{\text{adj}}$$

$$\csc\theta = \frac{c}{a} = \frac{\text{hyp}}{\text{opp}}$$

Example 13. Find $\sin\theta$, if $\cot\theta = \frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$.



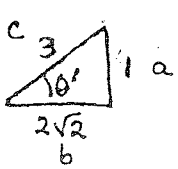
reference $\angle$ θ' with $\cot\theta' = \frac{3}{4}$

hyp $= \sqrt{4^2+3^2} = 5$

$\sin\theta' = \frac{4}{5}$

third quadrant, so $\sin\theta = -\frac{4}{5}$

Example 14. Find $\sec\theta$, if $\csc\theta = -3$ and $\frac{3\pi}{2} < \theta < 2\pi$



ref. angle θ' with $\csc\theta' = 3$

$b = \sqrt{3^2-1^2} = \sqrt{8} = 2\sqrt{2}$

$\sec\theta' = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$

fourth quadrant, so

$\sec\theta = \frac{3\sqrt{2}}{4}$

Exercises A.

① Draw the given angle in standard position: (a) 60° (b) 135° (c) 1140° (d) -240° (e) $\frac{\pi}{3}$ (f) $\frac{7\pi}{6}$ (g) $-\pi/6$ (h) $\frac{5\pi}{2}$

② Convert from degrees to radians: (a) 45° (b) 270° (c) 1° (d) -230° (e) 540°

③ Convert from radians to degrees: (a) $\frac{2\pi}{3}$ (b) 7π (c) $\frac{\pi}{6}$ (d) $\frac{19\pi}{2}$

④ Find the values of the following: (a) $\cos 5\pi$ (b) $\sin(-\frac{7\pi}{6})$ (c) $\cos\frac{23\pi}{4}$ (d) $\sin 9\pi$ (e) $\cos(-\frac{10\pi}{3})$ (f) $\sin(-\frac{4\pi}{3})$ (g) $\cot\frac{13\pi}{6}$ (h) $\tan\frac{9\pi}{2}$ (i) $\csc(-\frac{\pi}{6})$ (j) $\tan\frac{23\pi}{4}$ (k) $\sec\frac{10\pi}{3}$ (l) $\csc 5\pi$ (m) $\cot(-\frac{5\pi}{4})$ (n) $\sin 150^\circ$ (o) $\sec(-120^\circ)$ (p) $\csc 495^\circ$

⑤ Make a table (the heading and beginning of which are shown below) giving the values of all six trigonometric functions for $t = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}$, and 2π.

t	$\cos t$	$\sin t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
0	1	0	0	—	1	—
$\pi/6$	$\sqrt{3}/2$	$1/2$				
$\pi/4$						
⋮						

⑥ Find the value of

(a) $\cos t$, if $\tan t = -\frac{2}{3}$ and $\frac{3\pi}{2} < t < 2\pi$

(b) $\sin t$, if $\sec t = \frac{13}{5}$ and $0 < t < \frac{\pi}{2}$

(c) $\tan t$, if $\csc t = \frac{5}{3}$ and $\frac{\pi}{2} < t < \pi$

(d) $\cot t$, if $\csc t = \frac{5}{4}$ and $0 < t < \frac{\pi}{2}$

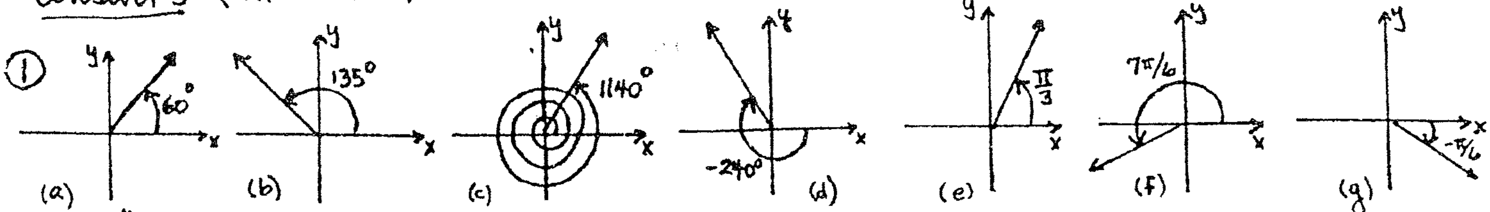
(e) $\sin\theta$, if $\cot\theta = -\frac{4}{9}$ and $\frac{\pi}{2} < \theta < \pi$

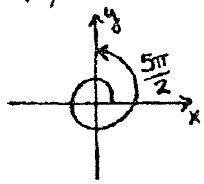
(f) $\cos\theta$, if $\tan\theta = \frac{\sqrt{3}}{2}$ and $\pi < \theta < \frac{3\pi}{2}$

(g) $\sec\theta$, if $\sin\theta = -\frac{1}{6}$ and $\frac{3\pi}{2} < \theta < 2\pi$

(h) $\csc\theta$, if $\cot\theta = -\frac{\sqrt{13}}{12}$ and $\frac{\pi}{2} < \theta < \pi$

Answers (Exercises A)

① 



② (a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{2}$ (c) $\frac{\pi}{180}$ (d) $-\frac{23\pi}{18}$ (e) 3π ③ (a) $120°$ (b) $1260°$ (c) $30°$ (d) $1710°$ ④ (a) -1 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{2}}{2}$ (d) 0 (e) $-\frac{1}{2}$ (f) $\frac{\sqrt{3}}{2}$ (g) $\sqrt{3}$ (h) DNE (i) -2 (j) -1 (k) -2 (l) DNE (m) -1 (n) $\frac{1}{2}$ (o) -2 (p) $\sqrt{2}$

⑤

t	$\cos t$	$\sin t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
0	1	0	0	—	1	—
$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/3$	$\sqrt{3}$	$2\sqrt{3}/3$	2
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$
$\pi/2$	0	1	—	0	—	1
$2\pi/3$	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$
$3\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$5\pi/6$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2
π	-1	0	0	—	-1	—
$7\pi/6$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}/3$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2
$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
$4\pi/3$	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}$	$\sqrt{3}/3$	-2	$-\frac{2\sqrt{3}}{3}$
$3\pi/2$	0	-1	—	0	—	-1
$5\pi/3$	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	2	$-\frac{2\sqrt{3}}{3}$
$7\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
$11\pi/6$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2
2π	1	0	0	—	1	—

⑥ (a) $\frac{3\sqrt{13}}{13}$ (b) $\frac{12}{13}$ (c) $-\frac{3}{4}$

(d) $\frac{3}{4}$ (e) $\frac{9\sqrt{97}}{97}$ (f) $-\frac{2\sqrt{7}}{7}$

(g) $\frac{6\sqrt{35}}{35}$ (h) $\frac{\sqrt{157}}{12}$

Section B Trigonometric Identities

I. Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

II. Double Angle Identity

$$\cos 2x = \cos^2 x - \sin^2 x$$

III. Functional Relationships

$$\frac{\sin x}{\cos x} = \tan x \qquad \frac{\cos x}{\sin x} = \cot x$$
$$\frac{1}{\cos x} = \sec x \qquad \frac{1}{\sin x} = \csc x$$

To show that a trigonometric equation is an identity, one must show that both sides of the proposed equality can be transformed into the same expression. It is most important to work with one side at a time. The idea is to simplify each side separately until the same expression is obtained in the two cases (or sides). [At no time are you to multiply both sides of the "=" by a constant. Work one side at a time!]

Example 1:

Show that $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$ is an identity.

$$\sec^2\theta + \csc^2\theta \stackrel{?}{=} \sec^2\theta\csc^2\theta$$

$$\begin{array}{c|c}
\dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} & \dfrac{1}{\cos^2\theta}\cdot\dfrac{1}{\sin^2\theta} \\[2ex]
\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta} & \dfrac{1}{\cos^2\theta\sin^2\theta} \\[2ex]
\dfrac{1}{\cos^2\theta\sin^2\theta} &
\end{array}$$

... is an identity

Therefore, since both sides of the equality can be transformed into the same expression, the original expression is an identity. When working with a proposed identity, consider the following suggestions:

1.) Simplify the more complicated side of the equation first.

2.) Find common denominators for sums and differences.

3.) Express all trigonometric functions in terms of sines and cosines and then simplify.

Example 2:

Show that $\sin t \cos t = \dfrac{1}{\tan t + \cot t}$ is an identity.

$$(\sin t)(\cos t) = \frac{1}{\tan t + \cot t}$$

$$\frac{1}{\dfrac{\sin t}{\cos t} + \dfrac{\cos t}{\sin t}}$$

$$\frac{1}{\dfrac{\sin^2 t + \cos^2 t}{\cos t \sin t}}$$

$$\frac{\cos t \sin t}{\sin^2 t + \cos^2 t}$$

$$\frac{\cos t \sin t}{1}$$

$$(\sin t)(\cos t)$$

... is an identity

Example 3: Show whether the following is an identity or not.

$$\frac{1 + \tan t}{\tan t} = \cot t + \sec^2 t - \tan^2 t$$

$$\frac{1}{\tan t} + \frac{\tan t}{\tan t} \qquad \cot t + (\tan^2 t + 1) - \tan^2 t$$

$$\cot t + 1 \qquad \cot t + 1$$

... is an identity

Example 4: Show whether $\dfrac{\tan^2 t - 1}{\sin t - \cos t} = \dfrac{\sin t - \cos t}{\cos^2 t}$ is or is not an identity.

$$\frac{\tan^2 t - 1}{\sin t - \cos t} = \frac{\sin t - \cos t}{\cos^2 t}$$

$$\frac{\left(\frac{\sin^2 t}{\cos^2 t} - 1\right)}{\sin t - \cos t}$$

$$\frac{\left(\frac{\sin^2 t - \cos^2 t}{\cos^2 t}\right)}{\sin t - \cos t}$$

$$\frac{\sin^2 t - \cos^2 t}{\cos^2 t} \cdot \frac{1}{\sin t - \cos t}$$

$$\frac{\cancel{(\sin t - \cos t)}(\sin t + \cos t)}{\cos^2 t\,\cancel{(\sin t - \cos t)}}$$

$$\frac{\sin t + \cos t}{\cos^2 t}$$

... not an identity

Since the right side does not equal the left for all possible values of t, the given equation is <u>not</u> an identity. Try $t = 0$ for example:

$$\frac{\sin(0) + \cos(0)}{[\cos(0)]^2} = \frac{0+1}{1} = 1; \quad \frac{\sin(0) - \cos(0)}{[\cos(0)]^2} = \frac{0-1}{1} = -1$$

$$1 \neq -1$$

Example 5: Show whether the following is or is not an identity:

$$\frac{\cos 2\theta}{\sin\theta + \cos\theta} = \cos\theta - \sin\theta$$

$$\frac{\cos^2\theta - \sin^2\theta}{\sin\theta + \cos\theta}$$

$$\frac{\cancel{(\cos\theta + \sin\theta)}(\cos\theta - \sin\theta)}{\cancel{(\sin\theta + \cos\theta)}}$$

$$\cos\theta - \sin\theta$$

... is an identity

Example 6: Show whether the following is or is not an identity:

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos^2\theta - \sin^2\theta$$

$$\cos^2\theta - (1 - \cos^2\theta)$$

$$\cos^2\theta - 1 + \cos^2\theta$$

$$2\cos^2\theta - 1$$

... is an identity

Example 7: Show whether the following is or is not an identity:

$$1 + \sin t = \cos^3 t \tan t + \sin^3 t$$

$$\cos^3 t \cdot \frac{\sin t}{\cos t} + \sin^3 t$$

$$\cos^2 t \sin t + \sin^3 t$$

$$\sin t(\cos^2 t + \sin^2 t)$$

$$(\sin t)(1)$$

$$\sin t$$

... not an identity

<u>Exercises B.</u> Show whether each of the following is or is not an identity:

① $\dfrac{\sin\theta}{\cos\theta} = 1 - \dfrac{\cos\theta}{\sin\theta}$

② $1-\cos^4\theta = (2-\sin^2\theta)\sin^2\theta$

③ $1-2\sin^2\theta = 2\cos^2\theta - 1$

④ $\dfrac{\sec\theta-\csc\theta}{\sec\theta+\csc\theta} = \dfrac{\tan\theta+1}{\tan\theta-1}$

⑤ $\dfrac{\sec^4 t - \tan^4 t}{1-2\tan^2 t} = 1$

⑥ $\sin^2\theta\cot^2\theta + \cos^2\theta\tan^2\theta = 1$

⑦ $\sec\theta - \dfrac{\cos\theta}{1+\sin\theta} = \cot\theta$

⑧ $\dfrac{\tan^2 x}{1+\cos x} = \dfrac{\sec x - 1}{\cos x}$

⑨ $(\csc t - \cot t)^2 = \dfrac{1-\cos t}{1+\cos t}$

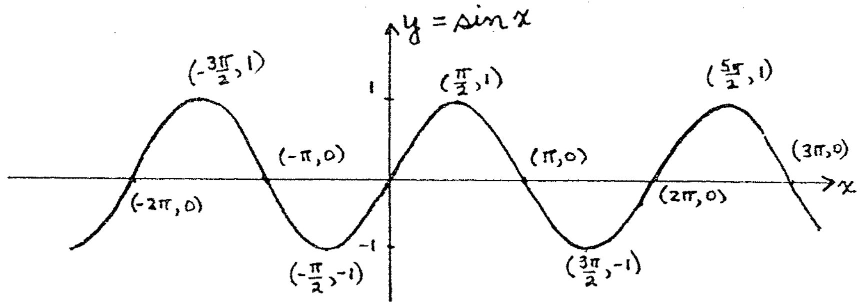
⑩ $1 + \dfrac{1}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$

<u>Answers</u> (Exercises B) The following are identities: ②, ③, ⑥, ⑧, ⑨, ⑩; the others are not identities.

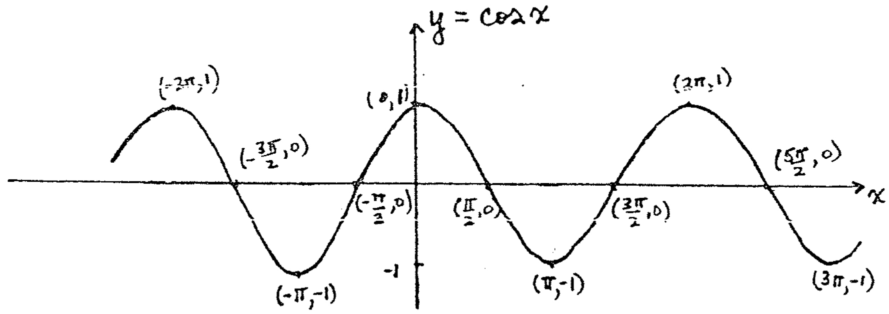
Section C. Graphs of Trigonometric Functions

I. Sine and Cosine Graphs

a. Basic Graphs



$y = \sin x$
amplitude = 1
period = 2π



$y = \cos x$
amplitude = 1
period = 2π

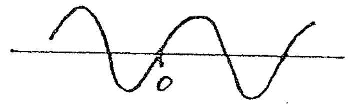
b. Graphs of the form $y = A\sin(bx+c)$ and $y = A\cos(bx+c)$

① amplitude = $|A|$
② period = $\frac{2\pi}{b}$
③ phase shift = $-\frac{c}{b}$
(set $bx+c=0$ & solve for x)

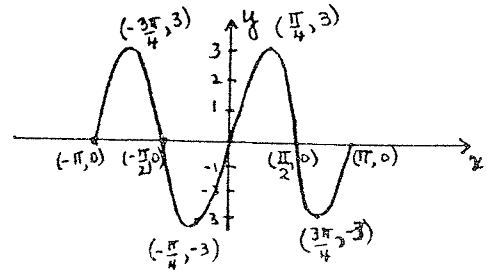
note also
- ④ sign of A (+ or −)
- ⑤ type (sine or cosine)
- ⑥ y-intercept ($x=0$)

Example 1. Sketch the graph of $y = 3\sin 2x$ for $-\pi \le x \le \pi$.

amp = 3
period = $\frac{2\pi}{2} = \pi$
c = 0 so no shift
+ sine type
$x=0 \to y = 3\sin 0 = 0$



$\longrightarrow$



Example 2. Sketch $y = 2\cos(x - \frac{\pi}{3})$ for $-2\pi \le x \le 2\pi$

amp = 2

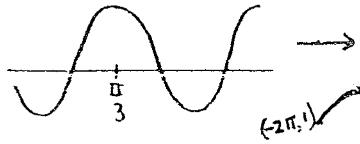
per = $\frac{2\pi}{1} = 2\pi$

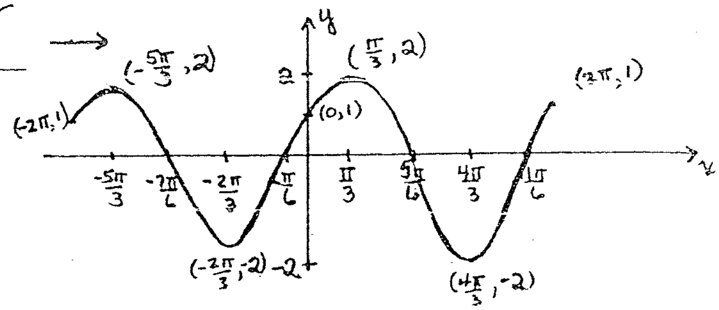
phase shift = $\frac{\pi}{3}$

+ cosine type

$x = 0 \to y = 2\cos(-\frac{\pi}{3})$

$= 2(\frac{1}{2}) = 1$





Example 3. Sketch $y = -\sin(2x + \frac{\pi}{2})$ for $-2\pi \le x \le 2\pi$.

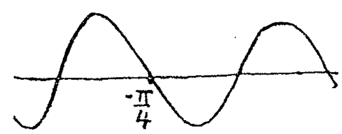
amp = 1

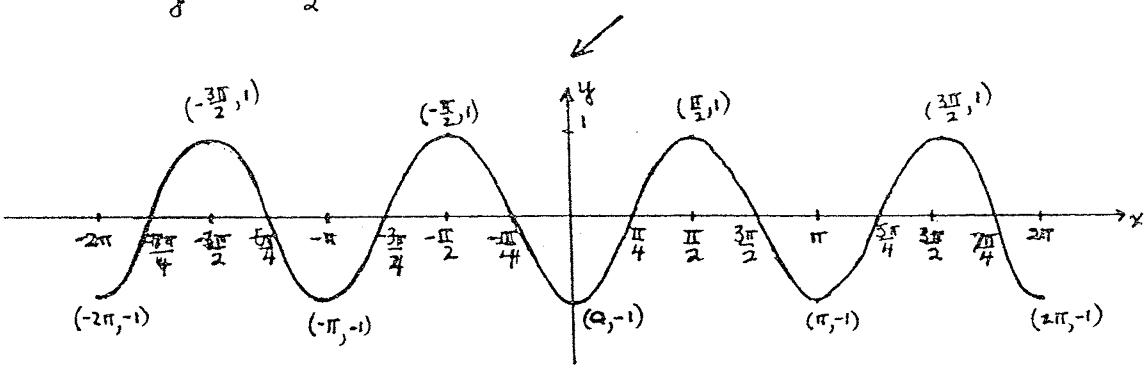
per = $\frac{2\pi}{2} = \pi$

phase shift = $-\pi/4$ $(2x + \frac{\pi}{2} = 0)$

− sine type

$x = 0 \to y = -\sin\frac{\pi}{2} = -1$





Example 4. Sketch $y = -3\cos(\frac{1}{2}x - \frac{\pi}{4})$ for $-3\pi \le x \le 3\pi$

amp = 3

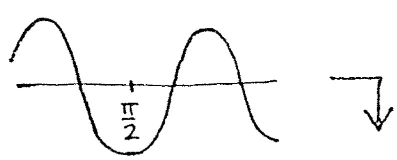
per = $\frac{2\pi}{1/2} = 4\pi$

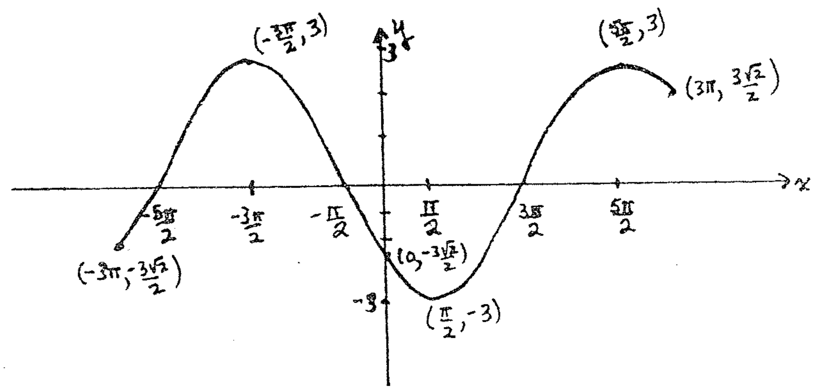
ph. shift = $\frac{\pi}{2}$

− cosine type

$x = 0 \to y = -3\cos(-\pi/4)$

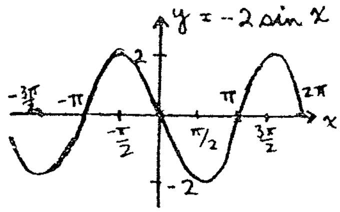
$= -3\sqrt{2}/2$

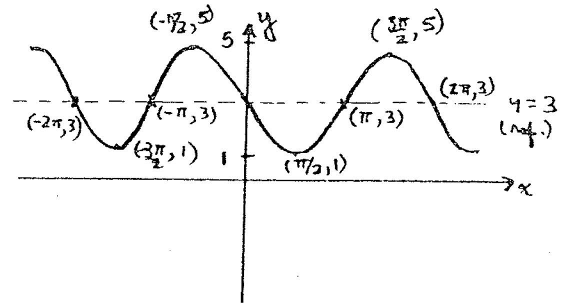




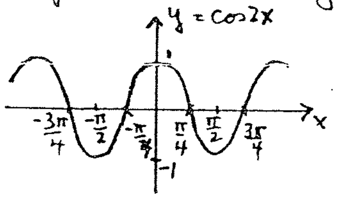
c. Some additional graphs (constant $\pm A \underset{\cos}{\sin}(bx+c)$)

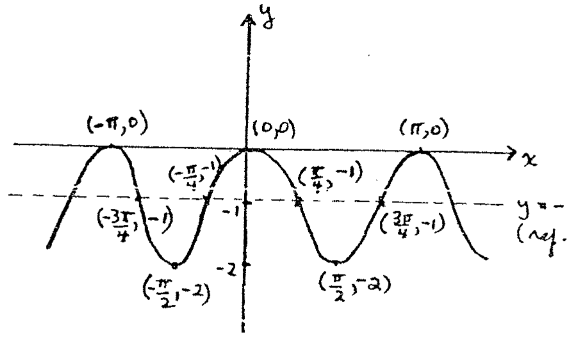
Example 5. Sketch $y = 3 - 2\sin x$





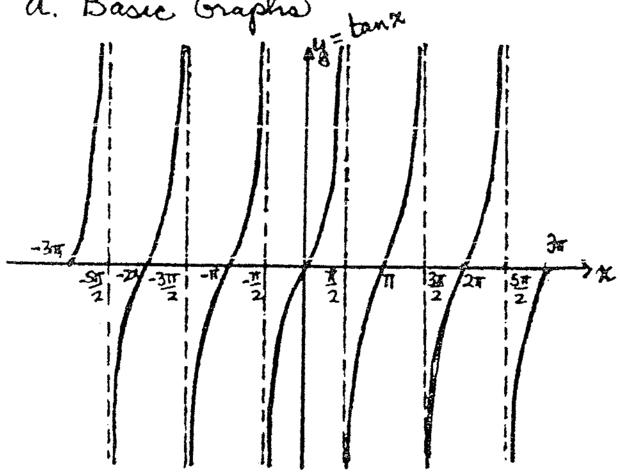
Example 6. Sketch $y = -1 + \cos 2x$





II. Graphs of Other Trigonometric Functions

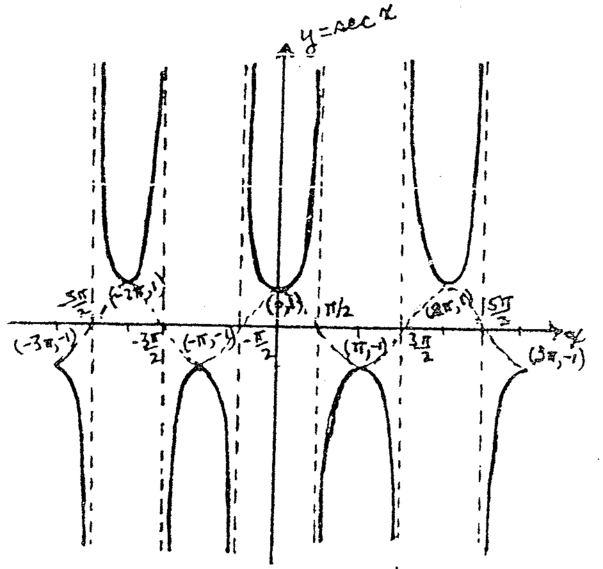
a. Basic Graphs



$y = \tan x = \dfrac{\sin x}{\cos x}$

period π ; asymptotes $x = \frac{\pi}{2} \pm n\pi$

intercepts $(\pm n\pi, 0)$ $n = 0, 1, 2, \dots)$

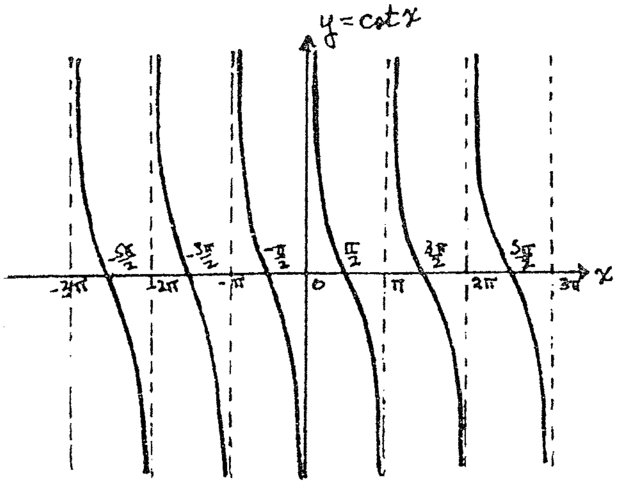


$y = \sec x = \dfrac{1}{\cos x}$

period 2π; asymptotes

$x = \frac{\pi}{2} \pm n\pi$, $n = 0, 1, 2, \dots$

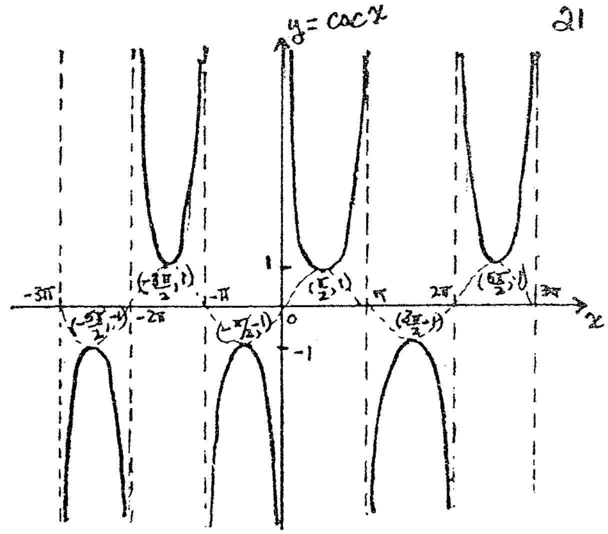
rel. max and min at $x = \pm n\pi$



$$y = \cot x = \frac{\cos x}{\sin x}$$

period π ; asymptotes $x = \pm n\pi$

intercepts $\left(\frac{\pi}{2} \pm n\pi, 0\right)$ $n = 0, 1, 2, \ldots$



$$y = \csc x = \frac{1}{\sin x}$$

period 2π ; asymptotes $x = \pm n\pi$

rel. max and min at

$$x = \frac{\pi}{2} \pm n\pi$$

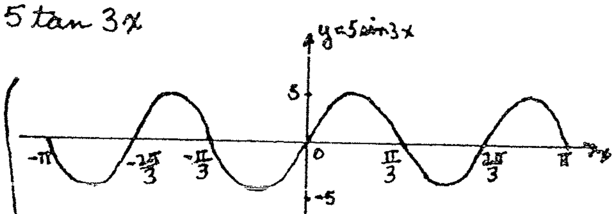
$n = 0, 1, 2, \ldots$

b. Examples of other graphs

<u>Example 7</u>. Sketch the graph of $y = 5 \tan 3x$

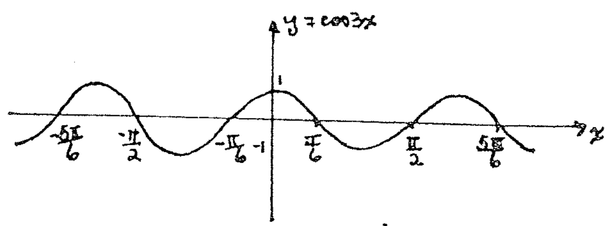
$$y = 5 \tan 3x = \frac{5 \sin 3x}{\cos 3x}$$

for reference



x- intercepts will be at $\pm \frac{n\pi}{3}$

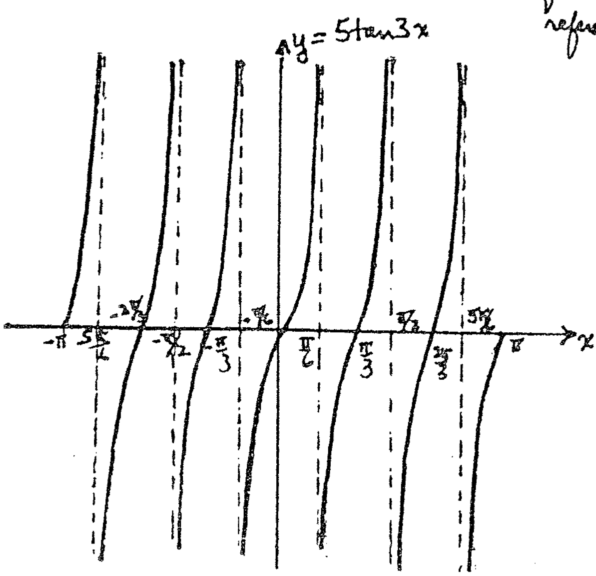
$n = 0, 1, 2, 3, \ldots$



asymptotes will be at

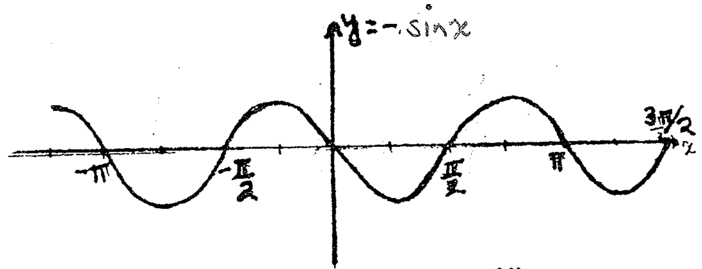
$$x = \frac{\pi}{6} \pm \frac{n\pi}{3}$$

$n = 0, 1, 2, \ldots$

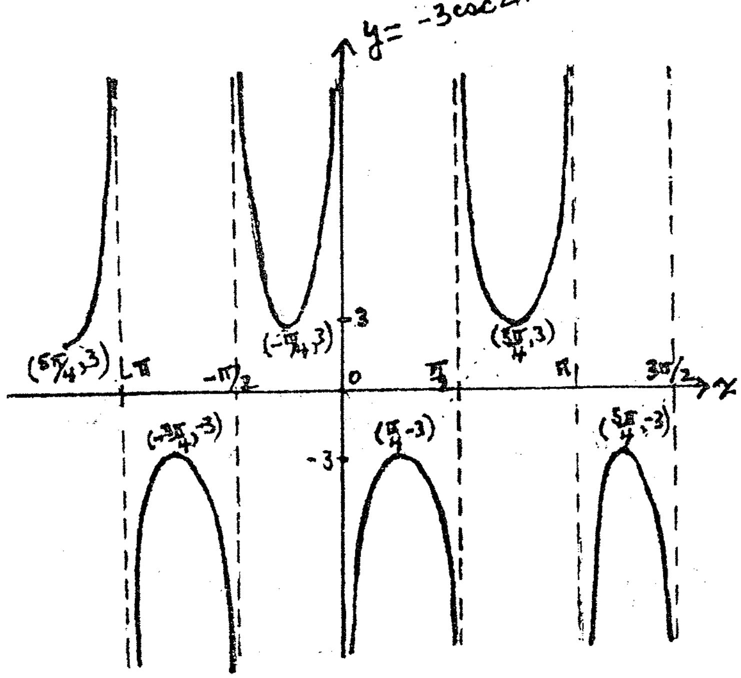


Example 8. Sketch the graph of $y = -3 \csc 2x$

$$y = -3 \csc 2x = \frac{3}{-\sin 2x}$$



for reference
asymptotes at $x = \pm \frac{n\pi}{2}$



Exercises C. ① Graph the following for $-2\pi \le x \le 2\pi$. Give the intercepts, amplitude, period, endpoints, and phase shift (when appropriate).

(a) $y = 4\cos x$ (b) $y = \sin \frac{2}{3}x$ (c) $y = 4\cos(2x - \frac{3\pi}{2})$ (d) $y = \sin(x - \frac{\pi}{6})$

(e) $y = -\frac{1}{2}\sin x$.

② Graph the following. Give the information asked for in ①. (a) $y = -\frac{8}{5}\cos(\frac{x}{5} + \frac{\pi}{3})$ over $[-5\pi, 10\pi]$ (b) $y = 4\sin(2x - \frac{\pi}{6})$ over $[-\pi, 2\pi]$ (c) $y = \frac{5}{2}\cos(2x + \frac{\pi}{4})$ from $-\pi$ to π (d) $y = \cos(x + \frac{\pi}{4})$ from -2π to 2π.

③ Graph the following. Label intercepts and other important points.

(a) $y = 1 + \cos x$ for $-2\pi \le x \le 2\pi$ (b) $y = 2 - \sin x$ from $-\pi$ to $\frac{3\pi}{2}$

(c) $y = 2 + 2\sin(\frac{x}{3} - \frac{\pi}{6})$ from $-\pi$ to 2π (d) $y = 2 - 3\cos 2x$ over $[-2\pi, \pi]$ (omit x-intercepts)

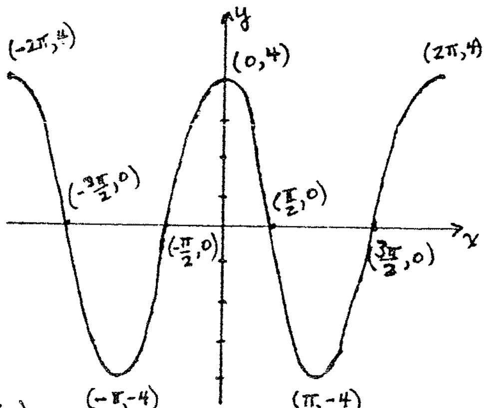
④ Graph the following. Label intercepts and other important features.

(a) $y = -\tan x$ over $[-2\pi, 2\pi]$ (b) $y = -\sec 2x$ from $-\pi$ to π

(c) $y = \frac{1}{2}\tan 2x$ from $-\frac{5\pi}{4}$ to $\frac{3\pi}{8}$ (d) $y = \csc 3x$ for $-\frac{\pi}{2} \le x \le \frac{5\pi}{6}$

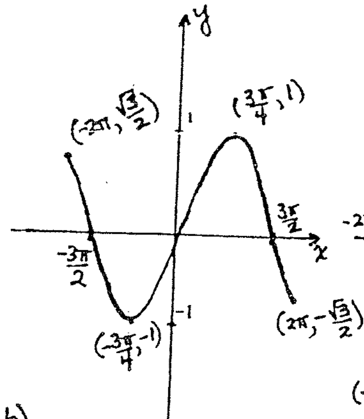
(e) $y = 2\tan\frac{x}{2}$ from -3π to $\frac{5\pi}{2}$ (f) $y = -\csc(4x + \pi)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Answers (Exercises C)



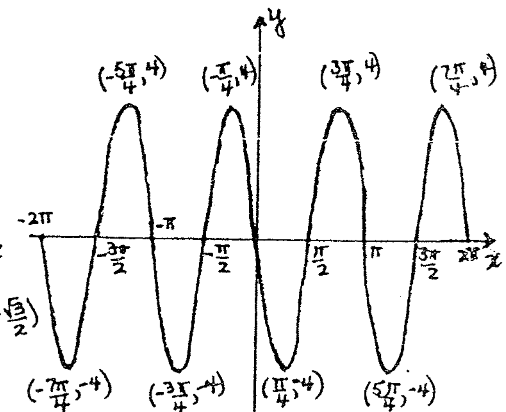
①(a)

period = 2π
amplitude = 4



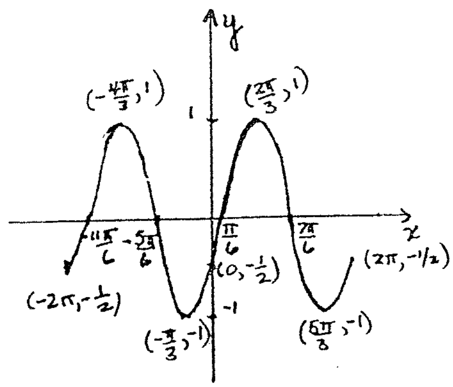
(b)

period = 3π
amplitude = 1

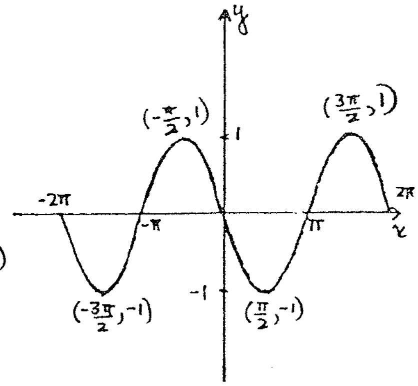


(c)

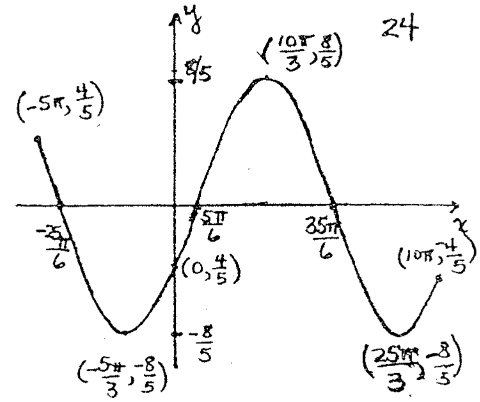
period = π
amplitude = 4
phase shift = $\frac{3\pi}{4}$ (to right)



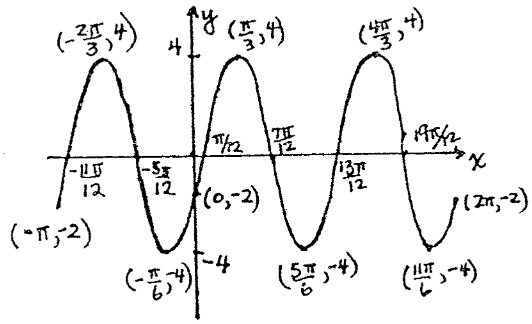
(d) period $= 2\pi$
amplitude $= 1$
phase shift $= \frac{\pi}{6}$



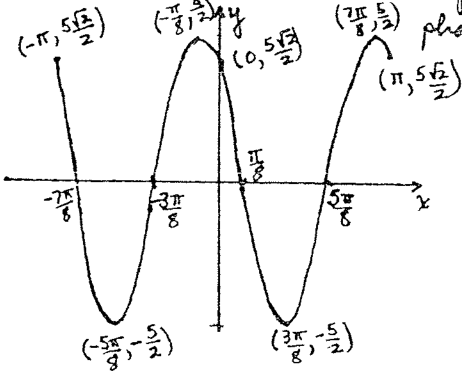
(e) period $= 2\pi$
amplitude $= \frac{1}{2}$



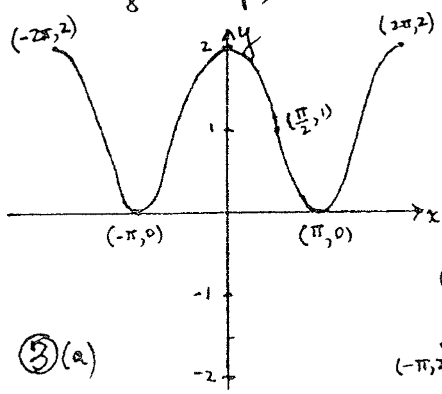
② (a) period $= 10\pi$
amplitude $= 8/5$
phase shift $= -\frac{5\pi}{3}$
(or $\frac{5\pi}{3}$ to the left)



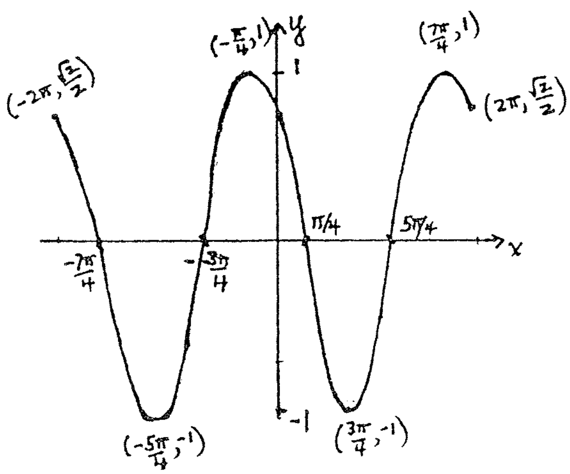
(b) period $= \pi$
amplitude $= 4$
phase shift $= \frac{\pi}{12}$ (right)



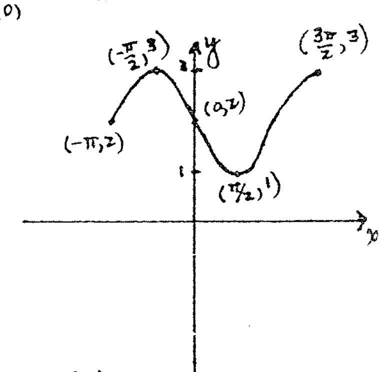
(c) period $= \pi$
amplitude $= \frac{5}{2}$
phase shift $= \frac{-\pi}{8}$
(or $\frac{\pi}{8}$ to the left)



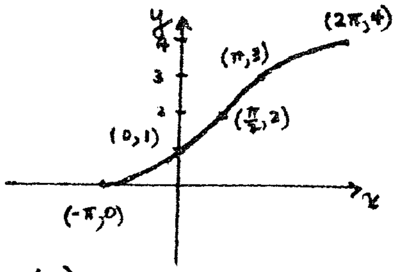
③ (a)



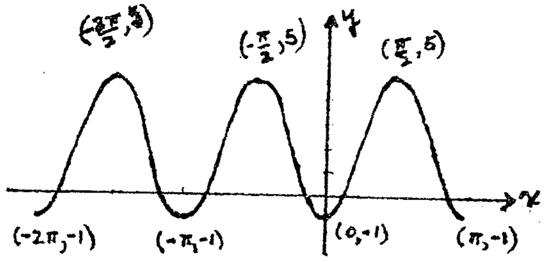
(d) period $= 2\pi$; amplitude $= 1$
phase shift $= -\pi/4$ ($\frac{\pi}{4}$ to left)



(b)

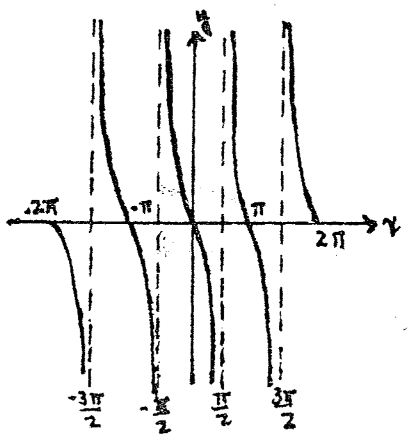

(2π,4)
(π,3)
(π/2,2)
(0,1)
(-π,0)

(c)

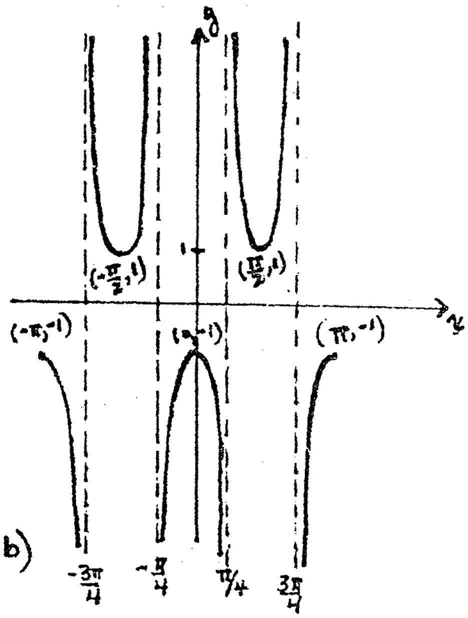

(-π/2,5)
(π/2,5)
(-2π,-1)
(-π,-1)
(0,-1)
(π,-1)

(d)

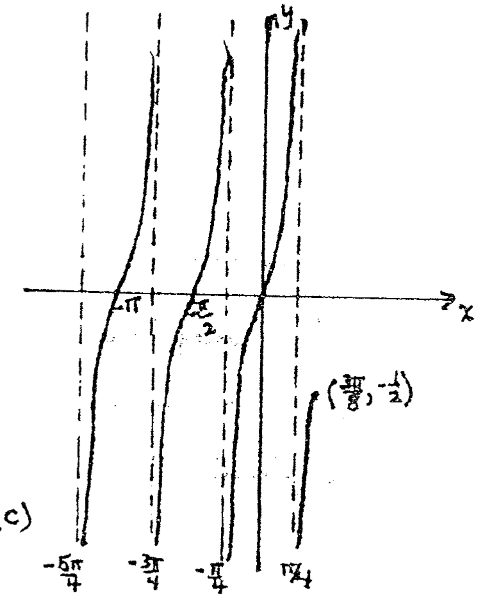
4 (a)


-2π
-π
π
2π
-3π/2
-π/2
π/2
3π/2

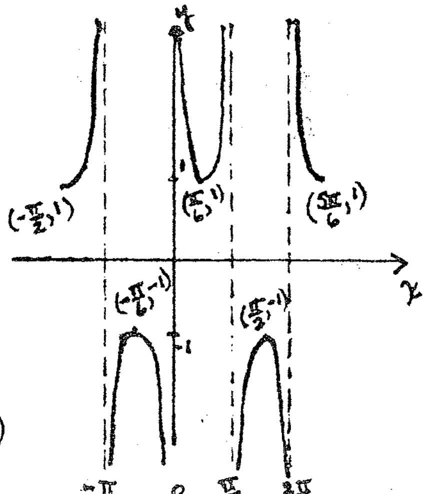
(b)


(-π/2,1)
(π/2,1)
(-π,-1)
(π,-1)
-3π/4
-π/4
π/4
3π/4

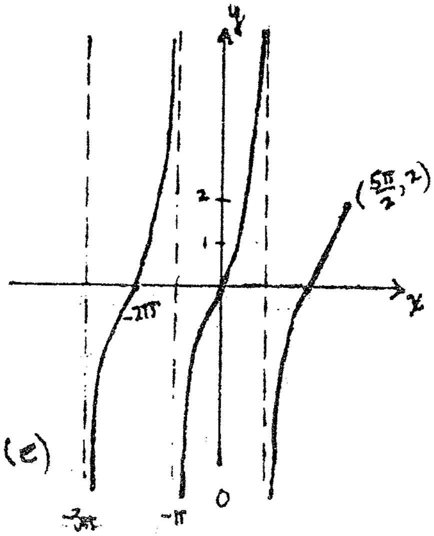
(c)


-π
-π/2
(3π/8, -1/2)
-5π/4
-3π/4
-π/4
π/4

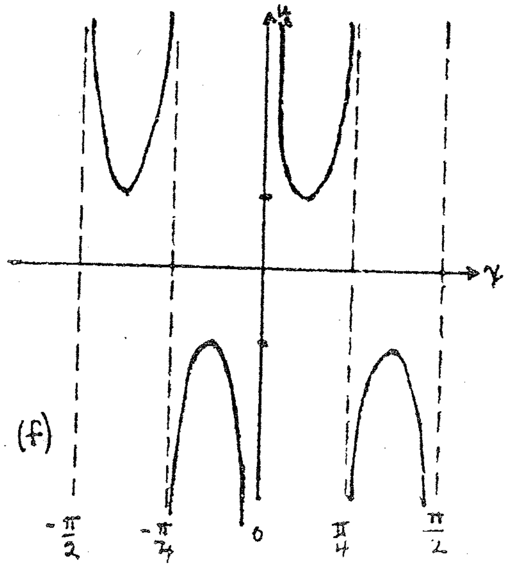
(d)


(-π/2,1)
(π/6,1)
(5π/6,1)
(-π/6,-1)
(π/2,-1)
-π/3
0
π/3
2π/3

(e)


(5π/2,2)
-2π
-3π
-π
0

(f)


-π/2
-π/4
0
π/4
π/2

Section D. Trigonometric Equations

When an equation contains a trigonometric expression with a variable such as $\sin x$, it is called a trigonometric equation. e.g.:

$$2 \sin x = 1 \qquad \text{or}$$

$$4 \cos^2 x - 8 \cos x + 3 = 0$$

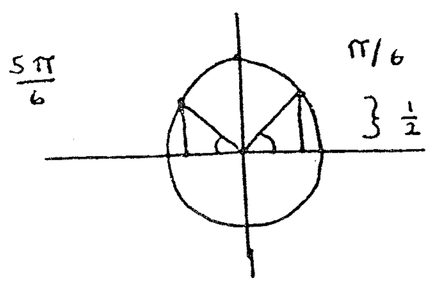
To solve each of these equations, we find all replacements for the variable that makes the equation true and there is an infinite number of replacements.

Example 1: Solve for all values of θ:

$$2 \sin \theta = 1$$

Solution:

$$\sin \theta = \frac{1}{2}$$



$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ for one counterclockwise revolution $(0 \leq \theta \leq 2\pi)$

Now, for all solution $(-\infty \leq \theta \leq \infty)$ then

$$\theta = \frac{\pi}{6} \pm 2\pi n \quad , \quad \frac{5\pi}{6} \pm 2\pi n$$

where n is the number of revolutions and 2π is the length in radians of one revolution on the unit circle. Also recall that the period of $\sin x$ is 2π and the values for $\sin x$ repeat every 2π radians.

Example 2: Solve for all values of x:

$$4\cos^2 x - 8\cos x + 3 = 0$$

Solution:

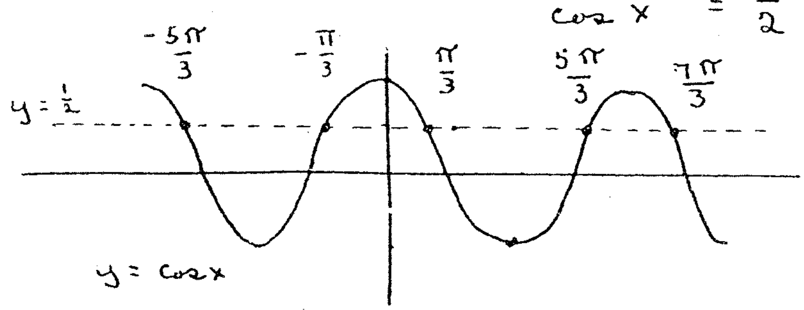
$$(2\cos x - 1)(2\cos x - 3) = 0$$

$$2\cos x - 1 = 0 \qquad 2\cos x - 3 = 0$$

$$2\cos x = 1 \qquad 2\cos x = 3$$

$$\cos x = \frac{1}{2} \qquad \cos x = \frac{3}{2}$$

No Solution; since $-1 \leq \cos x \leq 1$



$$x = \frac{\pi}{3} \pm 2n\pi, \quad \frac{5\pi}{3} \pm 2n\pi$$

where $n = 0, 1, 2, 3 \ldots$

Example 3: Solve for all values of x:

$$2\sin^2 x + (2 - \sqrt{3})\sin x - \sqrt{3} = 0$$

Solution:

$$(2\sin x - \sqrt{3})(\sin x + 1) = 0$$

$$2\sin x - \sqrt{3} = 0 \qquad \sin x + 1 = 0$$

$$\sin x = \frac{\sqrt{3}}{2} \qquad \sin x = -1$$

$$x = \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, \text{etc.} \qquad x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2} \text{ etc.}$$

$$x = \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}, \text{etc.}$$

Therefore, all solutions are:

$$x = \frac{\pi}{3} \pm 2\pi n,$$

$$\frac{2\pi}{3} \pm 2\pi n,$$

$$\frac{3\pi}{2} \pm 2\pi n$$

where $n = 0, 1, 2, 3 \ldots$

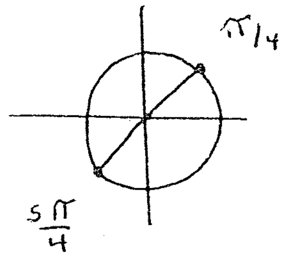
Example 4; Solve

$$\sin x = \cos x$$

Solution

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$



$$x = \frac{\pi}{4}, \frac{5\pi}{4}, \text{etc.}$$

Recall that the period of tangent x is π

therefore $x = \frac{\pi}{4} \pm \pi n$ where $n = 0, 1, 2, 3 \ldots$

Example 5: Solve

$$3(\cos^2\theta) - \cos 2\theta = 1$$

Solution.

$$3(\cos^2\theta) - (\cos^2\theta - \sin^2\theta) = 1$$

$$2(\cos^2\theta) + \sin^2\theta = 1$$

$$2(\cos^2\theta) + (1 - \cos^2\theta) = 1$$

$$\cos^2\theta + 1 = 1$$

$$\cos^2\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2} \pm 2\pi n, \quad \frac{3\pi}{2} + 2\pi n$$

where $n = 0, 1, 2, 3 \ldots$

Note: Since $\frac{\pi}{2} + \pi = \frac{3\pi}{2}$,

the solution can be written as

$$\theta = \frac{\pi}{2} \pm \pi n$$

where $n = 0, 1, 2, 3 \ldots$

Example 6 Solve

$$\sin x + \sqrt{\sin x} = 0$$

Note: Since we are taking the square root of sin x, the domain of the solution is restricted to $\sin x \geq 0$.

$$\sin x + \sqrt{\sin x} = 0$$

$$\sqrt{\sin x}\,(\sqrt{\sin x} + 1) = 0 \qquad \text{(factor)}$$

$$\sqrt{\sin x} = 0 \qquad\qquad \sqrt{\sin x} + 1 = 0$$

$$\sqrt{\sin x} = -1$$

$$\sin x = 0 \qquad\qquad \text{No Solutions}$$

$$x = 0 \pm 2\pi n,\ \pi \pm 2\pi n$$

therefore $x = 0 \pm \pi n$ where $n = 0, 1, 2 \ldots$

Example 7 Solve

$$\sec x \sin^2 x = \tan x$$

Caution: the domain is restricted to values $\cos x \neq 0$ since $\tan x$ is undefined where $\cos x = 0$.

Solution: $x = 0 \pm \pi n$ where $n = 0, 1, 2 \ldots$

Example 8 Solve

$$\cos x \sqrt{1 + \tan^2 x} = 1$$

Note: $1 + \tan^2 x = \sec^2 x$

Solution: all Real numbers, $x \neq \frac{\pi}{2} \pm \pi n$, $n = 0, 1, 2 \ldots$

$\tan x$ is undefined when $x = \frac{\pi}{2} \pm \pi n$.

Exercises D. Find all solutions of the following equations:

(1) $\tan x = 0$ (2) $2\cos x + \sqrt{2} = 0$ (3) $\cos^2 x - 1 = 0$

(4) $2\cos^2 x - 3\cos x - 2 = 0$ (5) $\tan^2 x + (\sqrt{3} - 1)\tan x - \sqrt{3} = 0$

(6) $3\sec^2 x = \sec x$ (7) $2\sin^2 x - \sin x - 1 = 0$ (8) $\cos 2x = \sin x$

(9) $\dfrac{1+\cos x}{\cos x} = 2$ (10) $\sqrt{\dfrac{1+2\sin x}{2}} = 1$

Answers (Exercises D)

(1) $0 \pm \pi n \quad (n = 0, 1, 2, \cdots)$

(2) $\left.\begin{matrix} \frac{3\pi}{4} \pm 2\pi n \\ \frac{5\pi}{4} \pm 2\pi n \end{matrix}\right\} (n = 0, 1, 2, \cdots)$

(3) $\left.\begin{matrix} 0 \pm 2\pi n \\ \pi \pm 2\pi n \end{matrix}\right\} (n = 0, 1, 2, \cdots)$ or, more compactly, $\pm \pi n \quad (n = 0, 1, 2, \cdots)$

(4) $\left.\begin{matrix} \frac{2\pi}{3} \pm 2\pi n \\ \frac{4\pi}{3} \pm 2\pi n \end{matrix}\right\} (n = 0, 1, 2, \cdots)$

(5) $\left.\begin{matrix} \frac{\pi}{4} \pm \pi n \\ \frac{2\pi}{3} \pm \pi n \end{matrix}\right\} (n = 0, 1, 2, \cdots)$

(6) no solutions

(7) $\left.\begin{matrix} \frac{7\pi}{6} \pm 2\pi n \\ \frac{11\pi}{6} \pm 2\pi n \\ \frac{\pi}{2} \pm 2\pi n \end{matrix}\right\} (n = 0, 1, 2, \cdots)$

(8) $\left.\begin{matrix} \frac{\pi}{6} \pm 2\pi n \\ \frac{5\pi}{6} \pm 2\pi n \\ \frac{3\pi}{2} \pm 2\pi n \end{matrix}\right\} (n = 0, 1, 2, \cdots)$

(9) $0 \pm 2\pi n \quad (n = 0, 1, 2, \cdots)$

(10) $\left.\begin{matrix} \frac{\pi}{6} \pm 2\pi n \\ \frac{5\pi}{6} \pm 2\pi n \end{matrix}\right\} (n = 0, 1, 2, \cdots)$

Section E <u>Inverse Trigonometric Functions</u>

I. Inverse functions

Definition: Two functions f and g are <u>inverses</u> of each other if and only if

$$f(g(x)) = x$$

for every value of x in the domain of g and

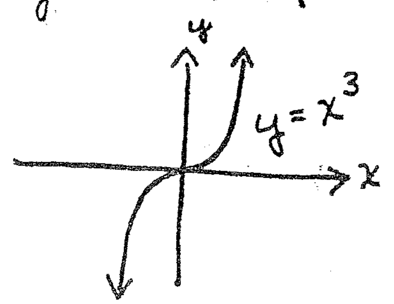
$$g(f(x)) = x$$

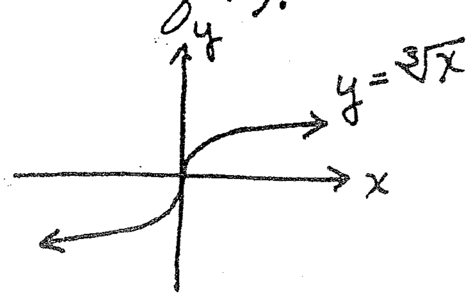
for every value of x in the domain of f.

Given $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$. f and g are inverses since

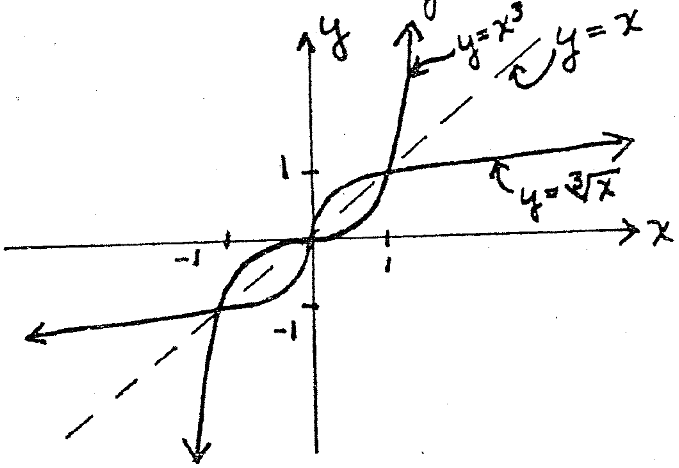
$$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x \text{ \underline{and}}$$
$$g(f(x)) = g(x^3) = \sqrt[3]{(x^3)} = x.$$

The inverse of a function can be found by reversing the values of the x and y-coordinates. If $f(x) = x^3$, then $(-2,-8)$, $(-1,-1)$, $(0,0)$, $(1,1)$, and $(2,8)$ are all on the graph of f. If the x and y-coordinates are reversed: $(-8,-2)$, $(-1,-1)$, $(0,0)$, $(1,1)$, and $(8,2)$, these points may be found on $g(x) = \sqrt[3]{x}$ (the inverse of f).





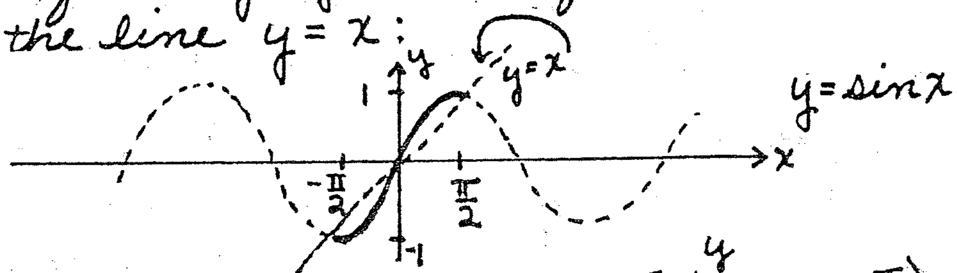
The graph of the inverse of a function can also be found by reflecting the original function about the line $y = x$.



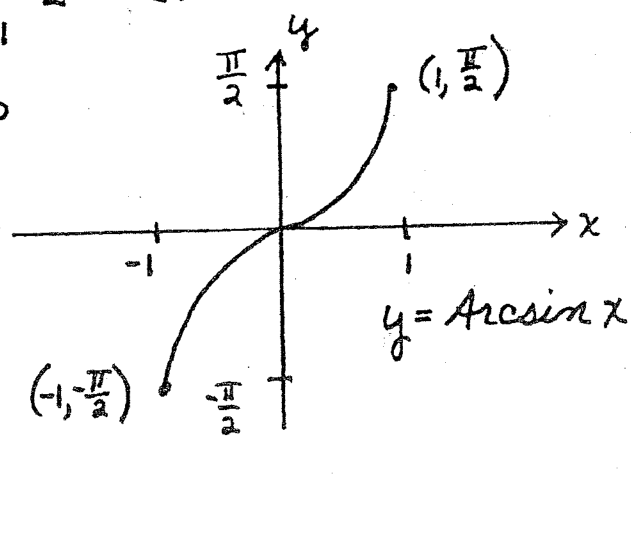
II. Trigonometric functions

The inverse for $y = \sin x$ is $y = \text{Arcsin}\, x$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

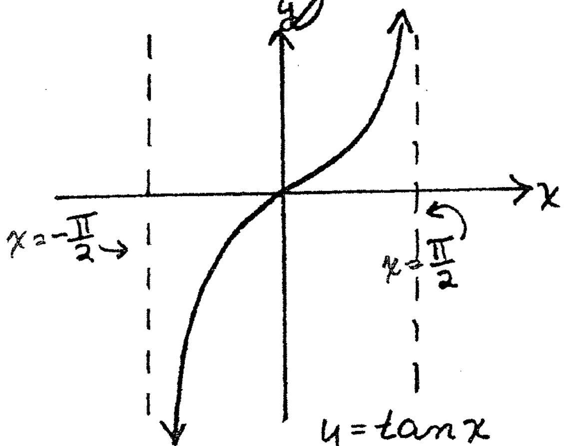
Note that $y = \text{Arcsin}\, x$ is the result of reflecting $y = \sin x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ about the line $y = x$:



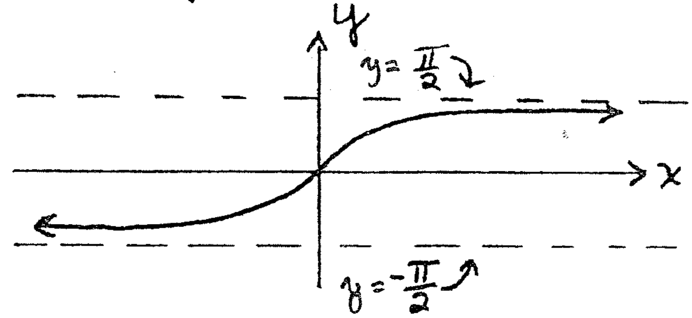
For $y = \text{Arcsin}\, x$ to be a function, the range must be limited to $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



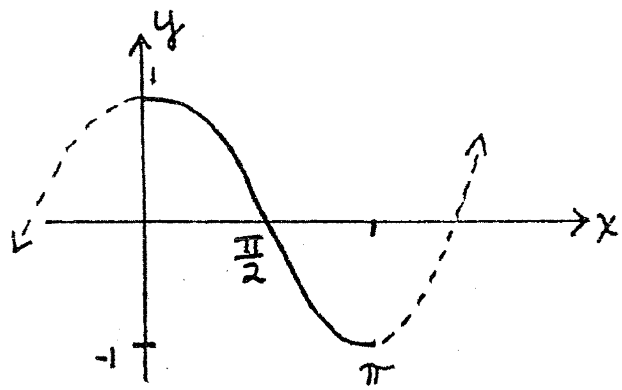
In like manner, the remaining five trigonometric functions have inverses, each with a defined limited range:



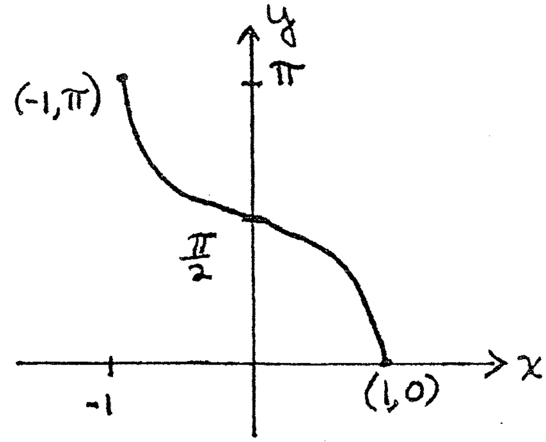
$y = \tan x$



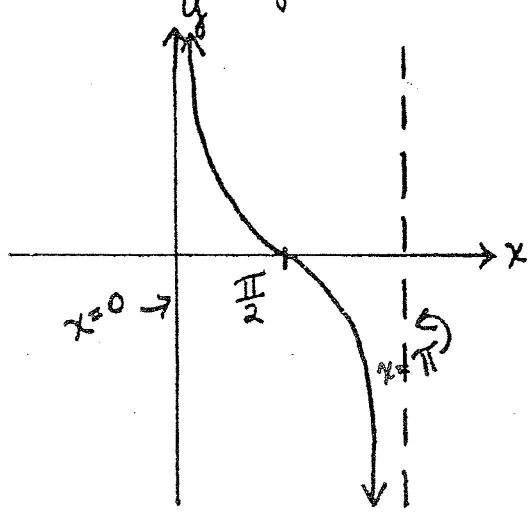
$y = \text{Arctan}\, x, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$



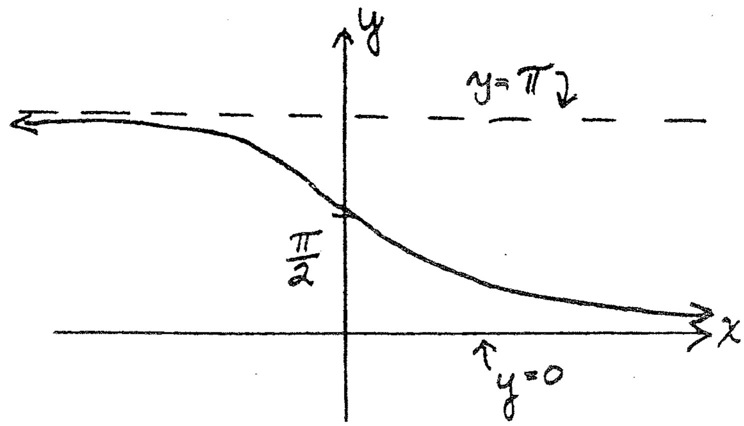
$y = \cos x$



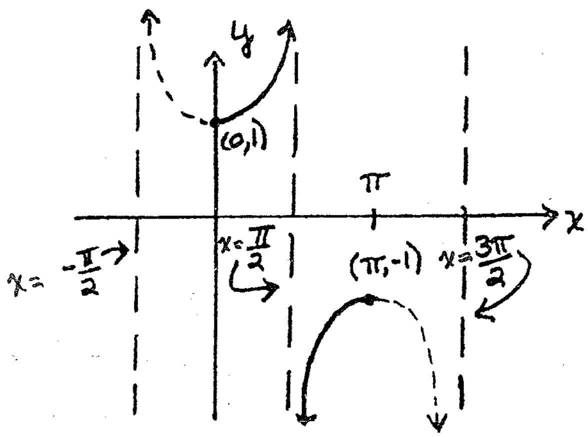
$y = \text{Arccos}\, x, \quad 0 \leq y \leq \pi$



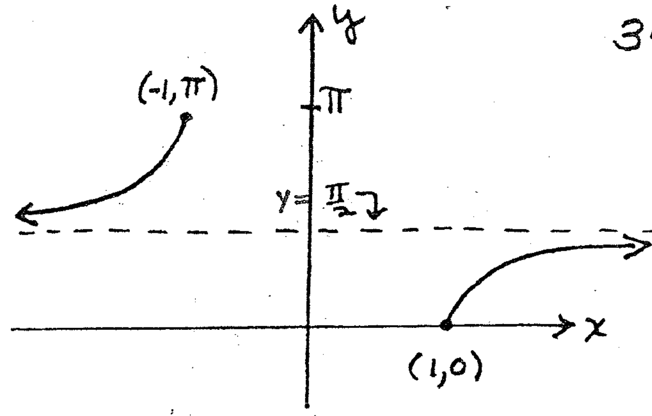
$y = \cot x$



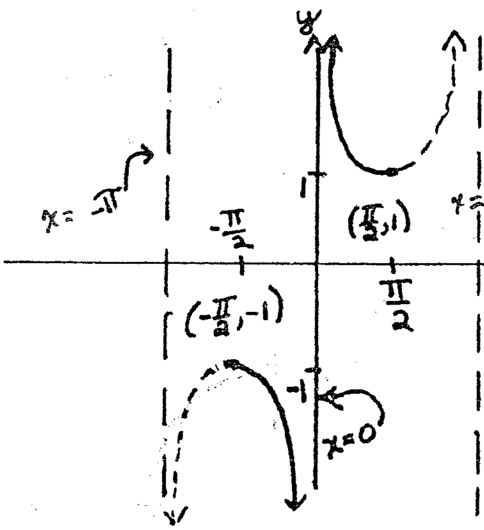
$y = \text{Arccot}\, x, \quad 0 < y < \pi$



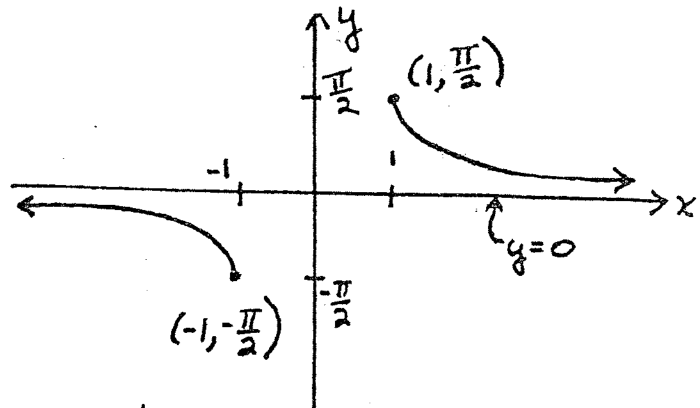
$y = \sec x$



$y = \text{Arcsec}\, x,\ 0 \le y < \frac{\pi}{2}$ and $\frac{\pi}{2} < y \le \pi$



$y = \csc x$



$y = \text{Arccsc}\, x,\ -\frac{\pi}{2} \le y < 0$ and $0 < y \le \frac{\pi}{2}$

III Examples

1. Find Arcsin 1

   Let $y = \text{Arcsin}\, 1$ where $\sin y = 1$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

   so that $y = \frac{\pi}{2}$

   $\text{Arcsin}\, 1 = \frac{\pi}{2}$

2. Find $\text{Arcsin}\left(-\frac{\sqrt{2}}{2}\right)$

   Let $y = \text{Arcsin}\left(-\frac{\sqrt{2}}{2}\right)$ where $\sin y = -\frac{\sqrt{2}}{2}$

   $y = -\frac{\pi}{4}$ in the range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

   $\text{Arcsin}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

3. Find $\sec(\text{Arcsec}\,\sqrt{2})$

Let $y = \text{Arcsec}\,\sqrt{2}$ where $\sec y = \sqrt{2}$

$y = \frac{\pi}{4}$ in the range $0 \le y < \frac{\pi}{2}$ and $\frac{\pi}{2} < y \le \pi$

$\sec(\text{Arcsec}\,\sqrt{2}) = \sec\frac{\pi}{4} = \underline{\underline{\sqrt{2}}}$

4. Find $\text{Arcsin}(\sin\frac{3\pi}{4})$

$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

Let $y = \text{Arcsin}(\sin\frac{3\pi}{4}) = \text{Arcsin}\left(\frac{\sqrt{2}}{2}\right)$

so that $\sin y = \frac{\sqrt{2}}{2}$ in the range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

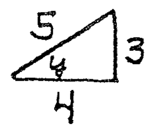
$\therefore y = \frac{\pi}{4}$ and $\text{Arcsin}(\sin\frac{3\pi}{4}) = \underline{\underline{\frac{\pi}{4}}}$

<u>NOTE</u>: Because of the limitations on the respective ranges of the inverse trigonometric functions, Arcsin (sin x) does not necessarily equal x (see example 4), Arctan(tan x) does not necessarily equal x, etc. However, sin (Arcsin x) = x, tan(Arctan x) = x, etc.

5. Find $\cos(\text{Arcsin}\,\frac{3}{5})$

Let $y = \text{Arcsin}\,\frac{3}{5}$ so that $\sin y = \frac{3}{5}$

and $\cos y = \frac{4}{5}$

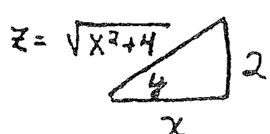


$\cos(\text{Arcsin}\,\frac{3}{5}) = \underline{\underline{\frac{4}{5}}}$

6. Find $\sin(\text{Arccot}\,\frac{x}{2})$

Let $y = \text{Arccot}\,\frac{x}{2}$ so that $\cot y = \frac{x}{2}$

$\sin y = \frac{2}{\sqrt{x^2+4}}$



$2^2 + x^2 = z^2$

$\sqrt{x^2+4} = z$

$\sin(\text{Arccot}\,\frac{x}{2}) = \frac{2}{\sqrt{x^2+4}}$

Exercises E.

① Find the values of the following: (a) $\text{Arcsin}\, 0$ (b) $\text{Arccot}\left(-\frac{\sqrt{3}}{3}\right)$

(c) $\cot(\text{Arccot}(-3))$ (d) $\cos\left(\text{Arccos}\,\frac{4}{5}\right)$ (e) $\text{Arccsc}\, 2$ (f) $\text{Arccos}(-1)$

(g) $\csc\left(\text{Arcsin}\,\frac{3}{5}\right)$ (h) $\text{Arcsec}\left(\sin\frac{\pi}{2}\right)$ (i) $\sin(\text{Arcsec}\, 2)$ (j) $\text{Arccos}\left(-\frac{1}{2}\right)$

(k) $\text{Arctan}^3(-\sqrt{3})$ (l) $3\,\text{Arcsin}^2\left(\frac{\sqrt{3}}{2}\right)$ (m) $\text{Arcsec}\, 0$ (n) $\sin(\text{Arctan}\, 2)$

(o) $\text{Arccos}\left(\sin\left(-\frac{\pi}{6}\right)\right)$ (p) $\tan\left(\text{Arccos}\left(\frac{-2}{3}\right)\right)$ (q) $\text{Arccos}\, 2$ (r) $\cos\left(\text{Arcsin}\left(-\frac{4}{5}\right)\right)$

(s) $4\,\text{Arctan}\, 1$ (t) $\csc(\text{Arcsec}\, 12)$ (u) $\text{Arccsc}\sqrt{2}$ (v) $\text{Arcsec}\, 2$

(w) $\text{Arctan}\left(\sin\frac{\pi}{2}\right)$ (x) $\text{Arctan}(\cos\pi)$ (y) $\text{Arcsin}\left(\tan\frac{\pi}{4}\right)$

② Write the given expression in terms of x without any trigonometric functions:

(a) $\sin(\text{Arctan}\, x)$ (b) $\tan(\text{Arcsin}\, x)$ (c) $\cot(\text{Arcsin}\, x)$

(d) $\cos(\text{Arcsin}\, x)$ (e) $\cos(\text{Arcsec}\, x)$ (f) $\csc\left(\text{Arccot}\,\frac{x}{4}\right)$

Answers (Exercises E)

① (a) 0 (b) $\frac{2\pi}{3}$ (c) -3 (d) $\frac{4}{5}$ (e) $\frac{\pi}{6}$ (f) π (g) $\frac{5}{3}$ (h) 0

(i) $\frac{\sqrt{3}}{2}$ (j) $\frac{2\pi}{3}$ (k) $-\pi^3/27$ (l) $\pi^2/3$ (m) no value (n) $\frac{2}{\sqrt{5}}$

(o) $\frac{2\pi}{3}$ (p) $-\sqrt{5}/2$ (q) no value (r) $\frac{3}{5}$ (s) π (t) $12/\sqrt{143}$

(u) $\pi/4$ (v) $\pi/3$ (w) $\pi/4$ (x) $-\pi/4$ (y) $\frac{\pi}{2}$

② (a) $\frac{x}{\sqrt{x^2+1}}$ (b) $\frac{x}{\sqrt{1-x^2}}$ (c) $\frac{\sqrt{1-x^2}}{x}$ (d) $\sqrt{1-x^2}$ (e) $\frac{1}{x}$

(f) $\frac{\sqrt{x^2+16}}{4}$

Review Exercises – Part I (Sections A – E)

① Find the values of the following:

(a) $\tan\left(-\frac{7\pi}{6}\right)$ (b) $\cot\theta$, if $\sin\theta = \frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$ (c) $\cos\left(-\frac{11\pi}{6}\right)$

(d) $\csc t$, if $\cos t = -\frac{15}{17}$ and $\pi < t < \frac{3\pi}{2}$ (e) $\tan(-\pi/6)$ (f) $\text{Arccos}\left(\sin\frac{11\pi}{4}\right)$

(g) $\cos(\text{Arctan}(-1))$ (h) $\sec\left(\text{Arccot}\left(\frac{-5}{12}\right)\right)$ (i) $\text{Arcsec}(-\sqrt{2})$ (j) $\sec\left(\text{Arccot}\frac{x}{3}\right)$

(k) $\text{Arctan}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$ (l) $\sec\left(-\frac{5\pi}{6}\right)$ (m) $9\,\text{Arccot}^2\frac{\sqrt{3}}{3}$ (n) $\text{Arccsc}(-2)$

(o) $\tan\left(\text{Arcsin}\left(-\frac{3}{5}\right)\right)$ (p) $\text{Arccos}\left(\cot\frac{11\pi}{4}\right)$ (q) $\csc t$, if $\tan t = 3$ and $\pi < t < \frac{3\pi}{2}$

(r) $\text{Arccsc}\left(\cos\left(-\frac{\pi}{3}\right)\right)$ (s) $\tan\theta$, if $\sin\theta = \frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$ (t) $\tan\frac{9\pi}{2}$

(u) $\cos\left(\text{Arctan}\frac{x}{3}\right)$ (v) $\text{Arcsin}\left(\cos\frac{\pi}{2}\right)$ (w) $\text{Arcsec}\ 1$ (x) $\csc\left(-\frac{9\pi}{4}\right)$ (y) $\cos(\text{Arcsin}\ x$

(z) $\tan\frac{54\pi}{6}$ (aa) $\text{Arcsec}^2\left(\csc\frac{2\pi}{3}\right)$ (bb) $\cos^2\frac{7\pi}{6}$ (cc) $\text{Arccsc}(-\sqrt{2})$

(dd) $5\,\text{Arccot}\left(\frac{-\sqrt{3}}{3}\right)$ (ee) $\cot\left(\text{Arcsin}\left(\frac{-8}{17}\right)\right)$ (ff) $\text{Arcsin}\left(\sin\frac{5\pi}{3}\right)$

(gg) $\cos(\text{Arctan}\ 5)$ (hh) $\text{Arccos}\left(\sin\left(-\frac{\pi}{6}\right)\right)$ (ii) $\sin t$, if $\cot t = \frac{-12}{5}$ and $\frac{3\pi}{2} < t$

(jj) $\cos\left(\text{Arctan}\left(-\frac{3}{4}\right)\right)$ (kk) $\text{Arcsec}\left(\frac{-2\sqrt{3}}{3}\right)$ (ll) $\text{Arccos}\left(\sin\frac{23\pi}{4}\right)$

(mm) $\cot t$, if $\sec t = \frac{-8}{5}$ and $\pi < t < \frac{3\pi}{2}$.

---

② Sketch graphs of the following. Label intercepts, asymptotes, and endpoints. On exercises marked with a *, give the amplitude, period, and phase shift.

*(a) $y = -4\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)$ on $[-\pi, 3\pi]$ *(b) $y = 4\sin\left(\frac{2}{3}x + \frac{\pi}{6}\right)$ from -2π to 2π

(c) $y = -2\sec(3x + \pi)$ on $\left[-\frac{\pi}{3}, \frac{2\pi}{3}\right]$ (d) $y = 2\csc\left(2x + \frac{\pi}{2}\right)$ for $-\frac{\pi}{2} \le x \le \pi$

*(e) $y = 5\cos\left(3x + \frac{\pi}{2}\right)$ on $[0, \pi]$ *(f) $y = 4 - 2\sin\left(\frac{x}{3} - \frac{\pi}{3}\right)$ from -2π to $\frac{5\pi}{2}$

(g) $y = \tan\left(x + \frac{\pi}{4}\right)$ on $[-\pi, 2\pi]$ *(h) $y = -2\sin\left(x - \frac{3\pi}{2}\right)$ on $\left[-\frac{\pi}{2}, 2\pi\right]$

(i) $y = \cot\left(x - \frac{\pi}{6}\right)$ from $-\pi$ to 2π *(j) $y = -\frac{1}{3}\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)$ on $[-2\pi, 4\pi]$

(k) $y = -\sec\frac{1}{2}x$ for $-3\pi < x < 3\pi$

③ Show whether each of the following is or is not an identity:

(a) $\dfrac{1}{\csc\theta + \cot\theta} + \dfrac{\sec\theta + 1}{\tan\theta} = 2\csc\theta$ (b) $\dfrac{1+\tan\theta}{1-\tan\theta} + \dfrac{1+\cot\theta}{1-\cot\theta} = 0$

(c) $\dfrac{1}{1+\cos\theta} - \dfrac{1}{1-\cos\theta} = \dfrac{2}{\sec\theta - \cos\theta}$ (d) $(1+\sec x)(1-\cos x) = \tan x \sin x$

(e) $\sec 2\theta = \dfrac{\sec^2\theta}{2-\sec^2\theta}$ (f) $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 2\sec\theta$

(g) $\dfrac{\tan^2 x \csc^2 x - 1}{\csc x \tan^2 x \sin x} = 1$ (h) $\dfrac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

(i) $\dfrac{\sin\theta + \tan\theta}{1+\cos\theta} = \cot\theta$ (j) $\cos^4 x - \sin^4 x = \cos 2x$

(k) $\dfrac{\tan t}{\tan^2 t - 1} = \dfrac{1}{\tan t - \cot t}$ (l) $\sec x - \sin x \tan x = \cos x$

(m) $\dfrac{\sin\theta + \cos\theta}{\sec\theta + \csc\theta} = \dfrac{\sin\theta}{\sec\theta}$ (n) $\dfrac{\cos^4 x - \sin^4 x}{\sin x + \cos x} = \dfrac{1-\tan x}{1+\tan x}$

④ Find all solutions of the following equations:

(a) $\tan^2 x + \sec^2 x + 3\sec x = 1$ (b) $\cos 2x = 1 + \sin x$

(c) $2\sin x \tan x + \tan x - 2\sin x - 1 = 0$ (d) $2\cos^2 x - 3\cos x + 1 = 0$

(e) $3\tan x + \dfrac{1}{\tan x} = 2\sqrt{3}$ (f) $2\sin x \tan x = 3$

(g) $\sec^2 x - \tan x = 1$ (h) $4\sin^2 x - 1 = 0$ (i) $\cos 2x = \cos x$

(j) $2\cos^3 x + \sin^2 x = 1$ (k) $8\sin^4 x - 10\sin^2 x + 3 = 0$

(l) $2\sin^2 x + 7\sin x + 3 = 0$ (m) $3\cot x = \tan x$

Answers (Review Exercises - Part I)

① (a) $-\frac{\sqrt{3}}{3}$ (b) $-\frac{\sqrt{5}}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $-\frac{17}{8}$ (e) $-\frac{\sqrt{3}}{3}$ (f) $\frac{\pi}{4}$ (g) $\frac{\sqrt{2}}{2}$ (h) $-\frac{13}{5}$ (i) $\frac{3\pi}{4}$

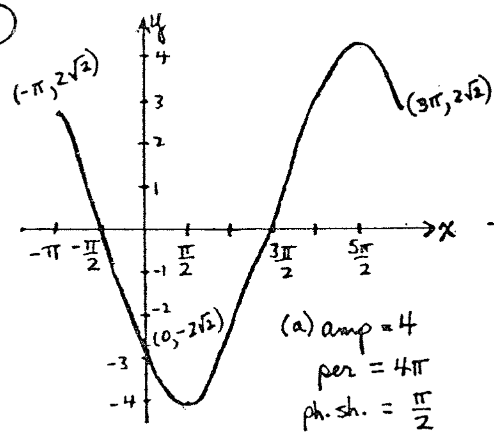
(j) $\frac{\sqrt{x^2+9}}{x}$ (k) $-\frac{\pi}{4}$ (l) $-\frac{2}{\sqrt{3}}$ (m) π^2 (n) $-\frac{\pi}{6}$ (o) $-\frac{3}{4}$ (p) π (q) $-\frac{\sqrt{10}}{3}$

(r) no value (s) $-\frac{5}{12}$ (t) no value (u) $\frac{3}{\sqrt{x^2+9}}$ (v) 0 (w) 0 (x) $-\sqrt{2}$

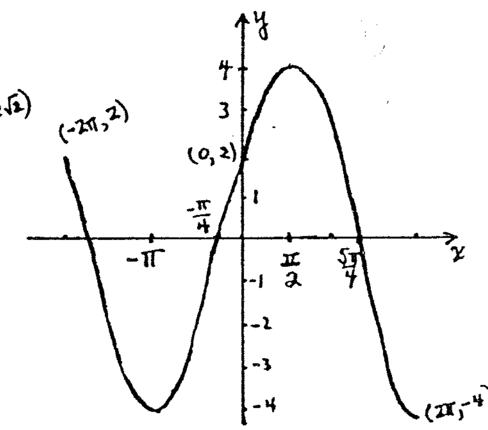
(y) $\sqrt{1-x^2}$ (z) 0 (aa) $\frac{\pi^2}{36}$ (bb) $\frac{3}{4}$ (cc) $-\frac{\pi}{4}$ (dd) $\frac{10\pi}{3}$ (ee) $-\frac{15}{8}$ (ff) $-\frac{\pi}{3}$

(gg) $\frac{\sqrt{26}}{26}$ (hh) $\frac{2\pi}{3}$ (ii) $-\frac{5}{13}$ (jj) $\frac{4}{5}$ (kk) $\frac{5\pi}{6}$ (ll) $\frac{\pi}{3}$ (mm) $\frac{5\sqrt{39}}{39}$

②

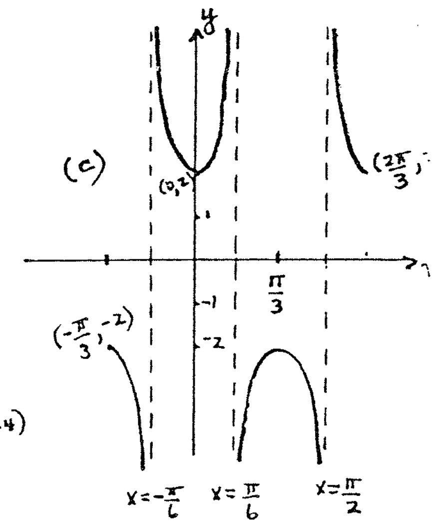


(a) amp = 4
per = 4π
ph. sh. = $\frac{\pi}{2}$

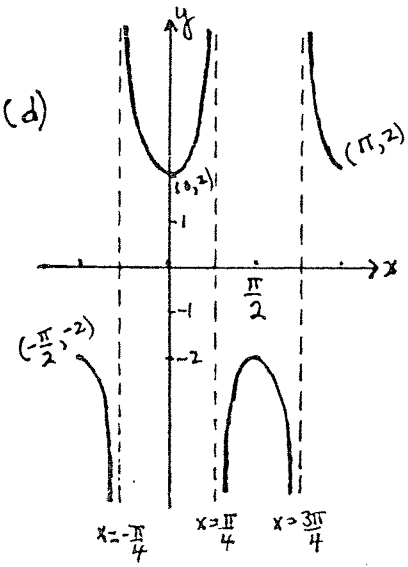


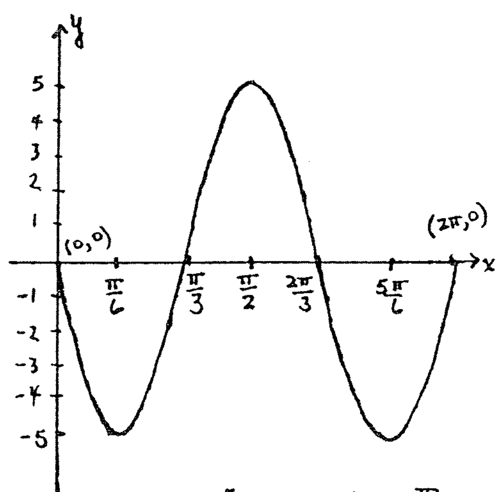
(b) amp = 4 ; per = 3π
ph. sh. = $-\frac{\pi}{4}$ ($\frac{\pi}{4}$ to left)

(c)

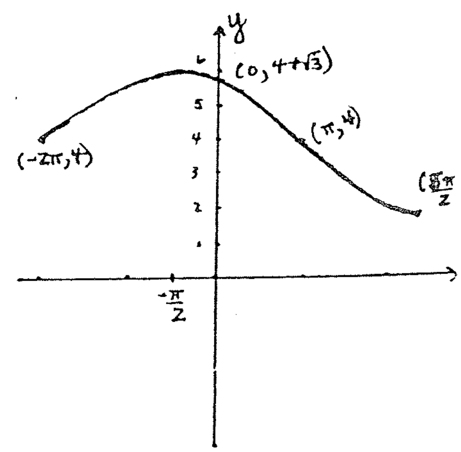


(d)



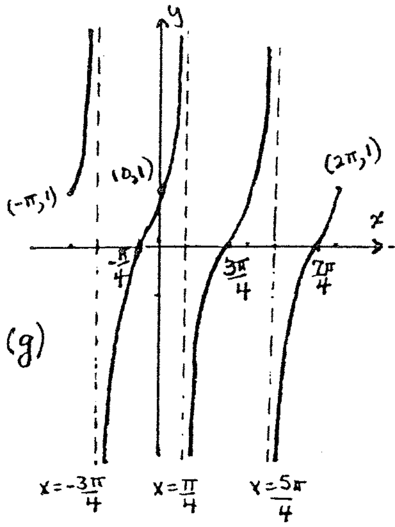


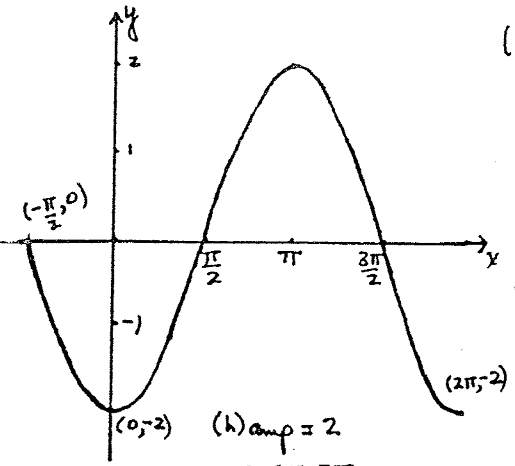
(e) amp. = 5
per. = $\frac{2\pi}{3}$
ph. sh. = $-\frac{\pi}{6}$
($\frac{\pi}{6}$ to left)



(f) amp = 2
period = 6π
ph. sh. = π

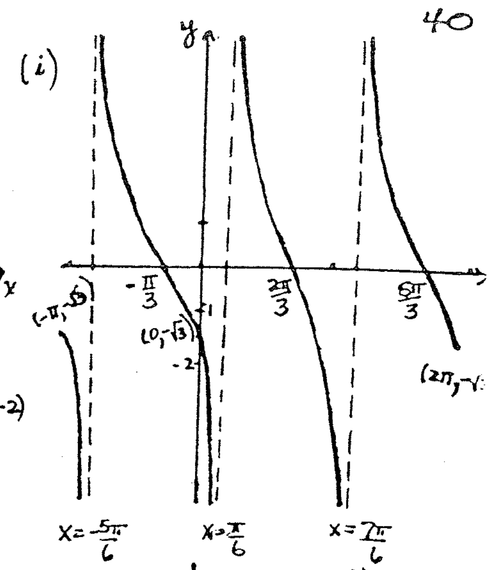
(g)

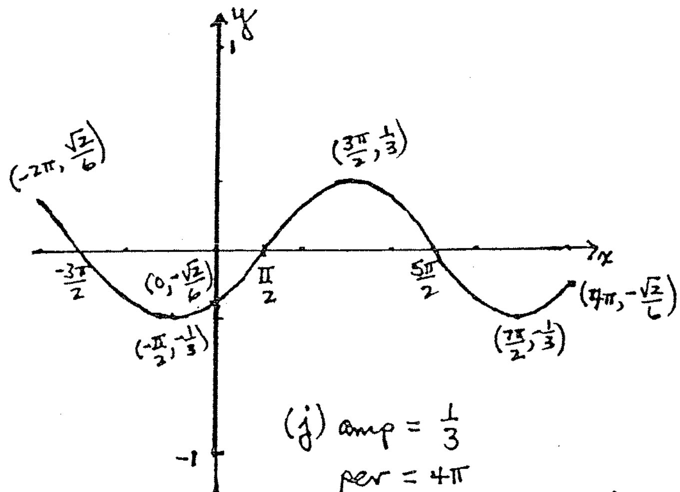




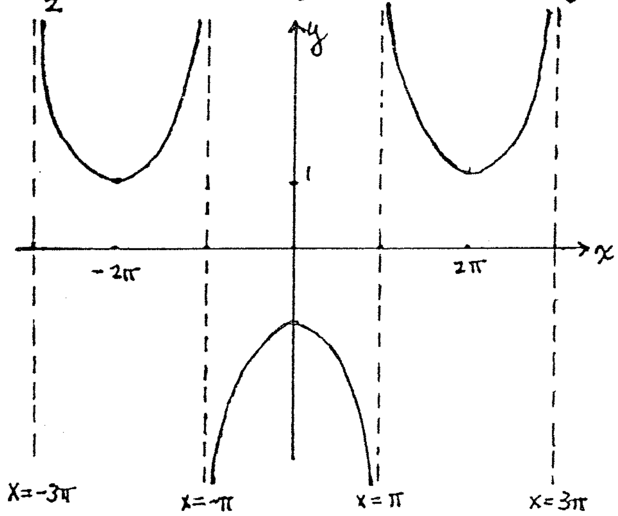
(h) amp = 2
per = 2π
ph sh. = $\frac{3\pi}{2}$

(i)





(j) amp = $\frac{1}{3}$
per = 4π
ph. sh. = $-\frac{\pi}{2}$ ($\frac{\pi}{2}$ to left)



③ (c), (i), (n) are not identities; the others are identities

④ The condition $n = 0, 1, 2, 3, \cdots$ should be attached to all answers below.

(a) $\frac{2\pi}{3} \pm 2\pi n$; $\frac{4\pi}{3} \pm 2\pi n$ (b) $0 \pm \pi n$; $\frac{7\pi}{6} \pm \pi n$; $\frac{11\pi}{6} \pm \pi n$

(c) $\frac{7\pi}{6} \pm 2\pi n$; $\frac{11\pi}{6} \pm 2\pi n$; $\frac{\pi}{4} \pm \pi n$ (d) $\frac{\pi}{3} \pm 2\pi n$; $\frac{5\pi}{3} \pm 2\pi n$; $0 \pm \pi n$

(e) $\frac{\pi}{6} \pm \pi n$ (f) $\frac{\pi}{3} \pm 2\pi n$; $\frac{5\pi}{3} \pm 2\pi n$ (g) $0 \pm \pi n$; $\frac{\pi}{4} \pm \pi n$

(h) $\frac{\pi}{6} \pm \pi n$; $\frac{5\pi}{6} \pm \pi n$ (i) $\frac{2\pi}{3} \pm 2\pi n$; $\frac{4\pi}{3} \pm 2\pi n$; $0 \pm 2\pi n$

(j) $\frac{\pi}{2} \pm \pi n$; $\frac{\pi}{3} \pm 2\pi n$; $\frac{5\pi}{3} \pm 2\pi n$ (k) $\frac{\pi}{4} \pm \pi n$; $\frac{3\pi}{4} \pm \pi n$; $\frac{\pi}{3} \pm \pi n$; $\frac{2\pi}{3} \pm \pi n$

(l) $\frac{7\pi}{6} \pm 2\pi n$; $\frac{11\pi}{6} \pm 2\pi n$ (m) $\frac{\pi}{3} \pm \pi n$; $\frac{2\pi}{3} \pm \pi n$

Section F. Exponential and Logarithmic Functions

I. Exponential Functions

A. Laws of Exponents

For any real number $b > 0$ and any real number x, the expression

$$b^x$$

is defined and represents a unique, positive real number. In addition, the laws of exponents hold; namely,

i) $b^0 = 1$

ii) $b^x \cdot b^y = b^{x+y}$

iii) $\dfrac{b^x}{b^y} = b^{x-y}$

iv) $(b^x)^y = b^{xy}$

v) $(ab)^x = a^x b^x$

vi) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

where $a > 0$, $b > 0$

x, y real

B. Exponential Function with Base b

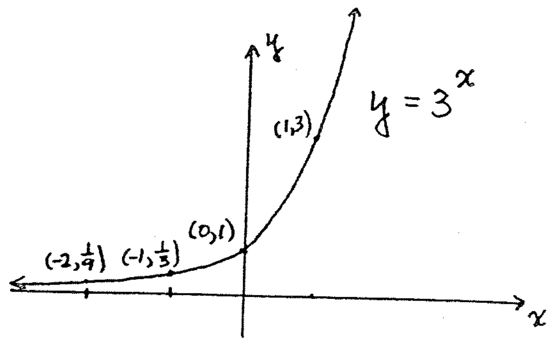
If $b > 0$ and $b \neq 1$, the exponential function base b is defined by

$$y = f(x) = b^x$$

Before stating the properties of exponential functions, we look at two examples.

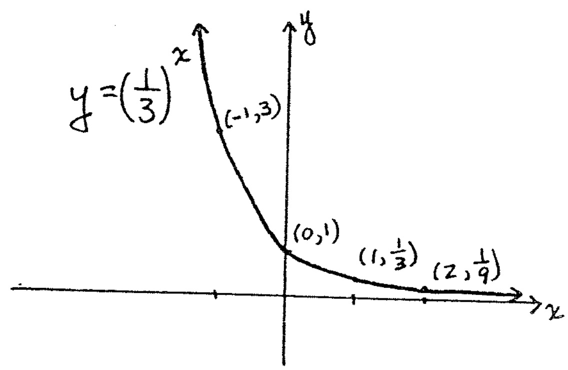
Example 1. Sketch the graph of $y = 3^x$

x	y
-3	$1/27$
-2	$1/9$
-1	$1/3$
0	1
1	3
2	9



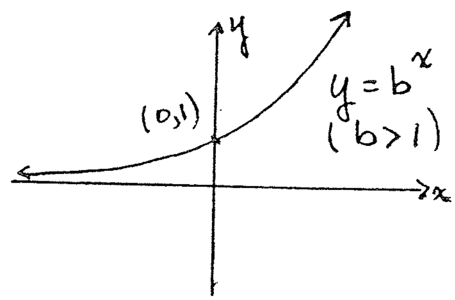
Example 2. Sketch the graph of $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

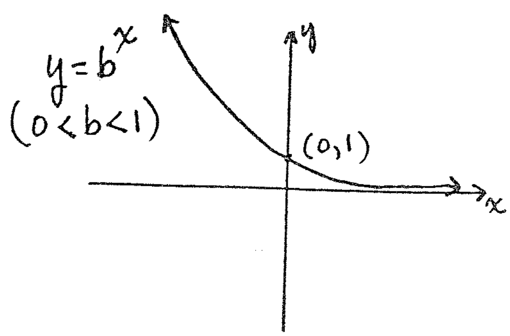
x	y
-3	27
-2	9
-1	3
0	1
1	$1/3$
2	$1/9$



Properties of Exponential Functions $y = b^x$

1. domain : all x
2. range : all $y > 0$
3. only intercept is $(0,1)$
4. $y = 0$ (x-axis) is a horizontal asymptote
5. function is increasing if $b > 1$
   function is decreasing if $0 < b < 1$
6. the function is one-to-one

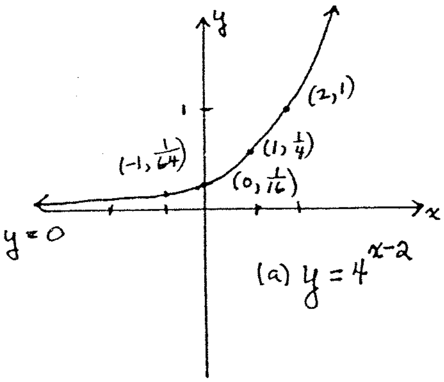




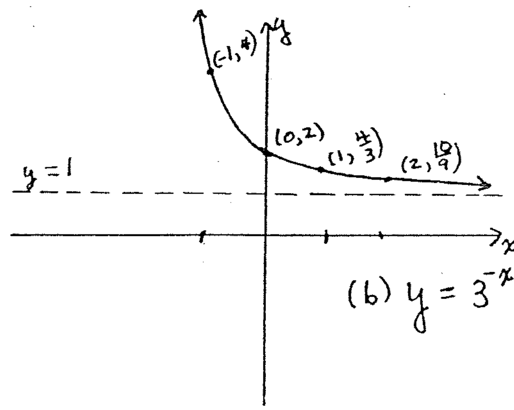
The following are more examples of exponential graphs.

Example 3. Sketch the graphs of (a) $y = 4^{x-2}$ (b) $y = 3^{-x} + 1$

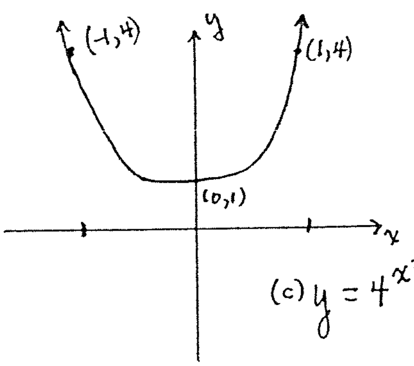
(c) $y = 4^{x^2}$ (d) $y = 4^{-x^2}$



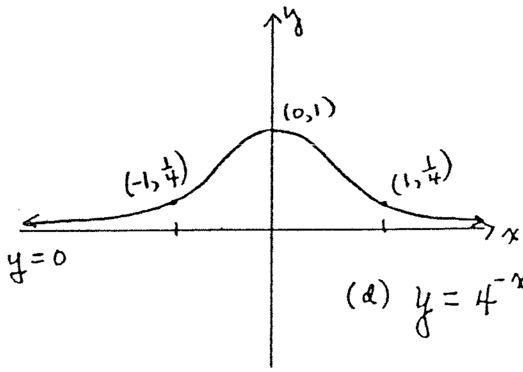
(a) $y = 4^{x-2}$



(b) $y = 3^{-x} + 1$



(c) $y = 4^{x^2}$



(d) $y = 4^{-x^2}$

C. The Exponential Function With Base e

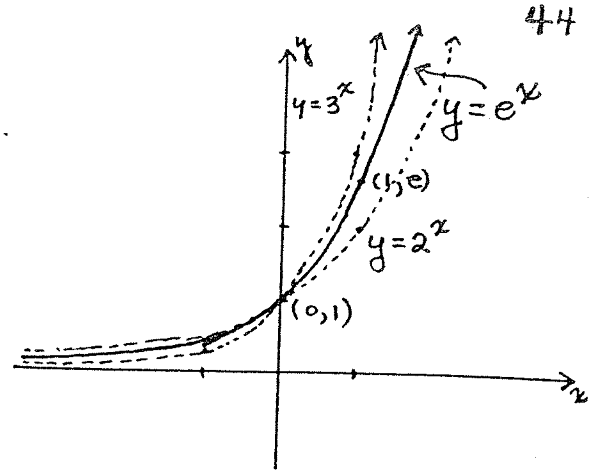
The irrational number e occurs frequently in the mathematical analysis of physical, sociological, and economic phenomena. The number e is given approximately by

$$e \approx 2.7182818$$

and is the most natural choice for the base of an exponential function:

$$y = e^x$$

The graph of this "natural" exponential function is shown at right. Graphs of $y = 2^x$ and $y = 3^x$ are drawn (dotted) for comparison.



II. Logarithmic Functions

A. Definition

Since the exponential function $y = b^x$ (where $b > 0$, $b \neq 1$) is one-to-one, the inverse relation

$$x = b^y$$

defines y as a function of x. The inverse function is written

$$y = \log_b x$$

and we say that y is the logarithm to the base b of x

Thus,

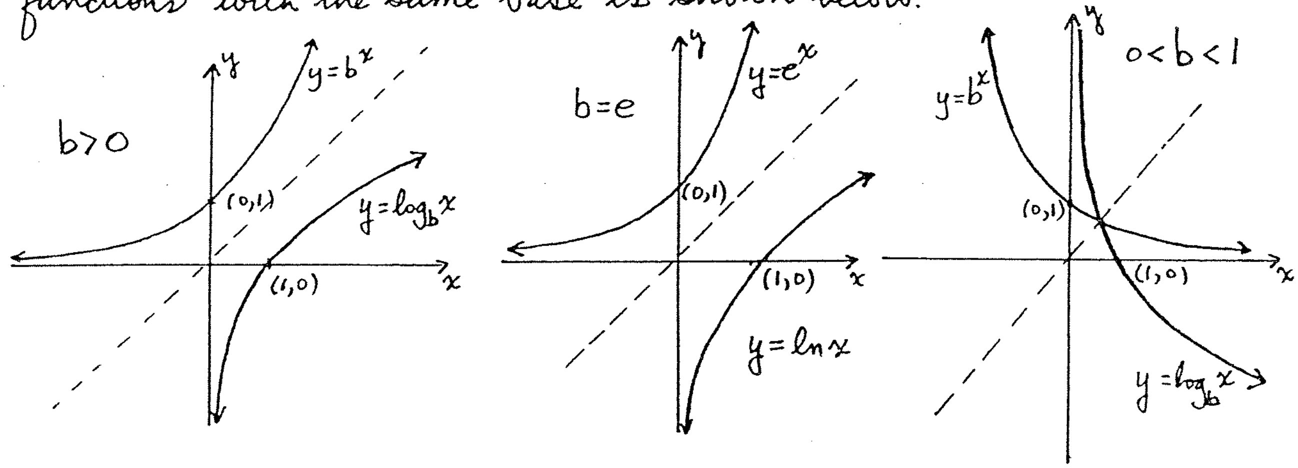
$$y = \log_b x \text{ if and only if } b^y = x$$

In the special case when $b = e$ we write $y = \ln x$ ($\equiv \log_e x$) and say that y is the natural logarithm of x. Thus

$$y = \ln x \text{ if and only if } e^y = x.$$

B. Properties of Logarithmic Functions & Their Graphs

The relationship between graphs of logarithmic and exponential functions with the same base is shown below.



Properties of $y = \log_b x$

1. domain: $x > 0$
2. range: all y
3. the only intercept is $(1,0)$
4. the line $x=0$ (y-axis) is an asymptote
5. if $b>1$, the function is increasing on $(0,\infty)$
   (this statement would apply to $y = \ln x$)
6. the function is one-to-one

Because of the relationship between logarithmic and exponential functions we can write any exponential form $b^y = x$ as an equivalent logarithmic form, and vice versa. Examples are given below

$\log_3 9 = 2 \leftrightarrow 3^2 = 9$

$\log_{10} 0.00001 = -5 \leftrightarrow 10^{-5} = 0.00001$

$\log_8 4 = \frac{2}{3} \leftrightarrow 8^{2/3} = 4$

$\log_{1/2} 4 = -2 \leftrightarrow \left(\frac{1}{2}\right)^{-2} = 4$

$\log_b \sqrt{b} = \frac{1}{2} \leftrightarrow b^{1/2} = \sqrt{b}$

$\ln 1 = 0 \leftrightarrow e^0 = 1$

$\log_3 C = N \leftrightarrow 3^N = C$

$\ln 4 = x \leftrightarrow e^x = 4$

Example 7. Find the values of the following: (a) $\log_4 8$ (b) $\log_{1/9}\sqrt{27}$ (c) $\ln \sqrt[3]{e^2}$

Solutions.

(a) let $y = \log_4 8$

$4^y = 8$ (exp. form)

$(2^2)^y = 2^3$ (common base)

$2^{2y} = 2^3$ (simplify)

$2y = 3$ (equate exponents One-one)

$\boxed{y = \frac{3}{2}}$

(b) let $y = \log_{1/9}\sqrt{27}$

$(\frac{1}{9})^y = (27)^{1/2}$

$(3^{-2})^y = (3^3)^{1/2}$

$3^{-2y} = 3^{3/2}$

$-2y = \frac{3}{2}$

$\boxed{y = -\frac{3}{4}}$

(c) let $\ln \sqrt[3]{e^2} = y$

$e^y = \sqrt[3]{e^2} = e^{2/3}$

$\boxed{y = \frac{2}{3}}$

Example 8. Solve for the unknown: (a) $\log_b 16 = \frac{4}{3}$ (b) $\ln x = -1$

Solutions.

(a) $\log_b 16 = \frac{4}{3}$

$b^{4/3} = 16$ (exp. form)

$b = (16)^{3/4}$ (solve)

$\boxed{b = 8}$ (simplify)

note: $b > 0$

(b) $\ln x = -1$

$e^{-1} = x$ (exp. form)

$\boxed{x = \frac{1}{e}}$

C. Further Properties of Logarithms

The following properties follow from the inverse relationship:

i) $\log_b b = 1$ $\qquad\qquad \ln e = 1$

ii) $\log_b 1 = 0$ $\qquad\qquad \ln 1 = 0$

iii) $\log_b b^x = x$ $\qquad\qquad \ln e^x = x$

iv) $b^{\log_b x} = x$ $\qquad\qquad e^{\ln x} = x$

$\Bigg\}$ special case $b = e$

These additional properties follow from the laws of exponents:

v) $\log_b xy = \log_b x + \log_b y$ $\qquad\qquad \ln xy = \ln x + \ln y$

vi) $\log_b \dfrac{x}{y} = \log_b x - \log_b y$ $\qquad\qquad \ln \dfrac{x}{y} = \ln x - \ln y$

vii) $\log_b x^n = n \log_b x$ $\qquad\qquad \ln x^n = n \ln x$

<u>NOTE</u>. <u>None</u> of these properties allows us to simplify such expressions as $\log_b(x+y)$; $(\log_b x)(\log_b y)$; $(\log_b x)^n$; $\dfrac{\log_b x}{\log_b y}$

<u>Example 9</u>. Simplify: (a) $2^{\log_2 5x}$ $\qquad$ (b) $\ln\left(\dfrac{e^5}{e^3}\right)$ $\qquad$ (c) $\log_5(3-2)$

<u>Solutions</u>

(a) $2^{\log_2 5x} = 5x$

(by prop. (iv))

(b) $\ln\left(\dfrac{e^5}{e^3}\right) = \ln e^2$ (laws of exp.)

$= 2$ (by prop. (iii))

(c) $\log_5(3-2) = \log_5 1$

$= 0$ (by prop. ii)

Example 10. Write as a single logarithm (a) $\ln 3 + 2\ln 4$
(b) $\frac{1}{2}\log_5 49 - \frac{1}{3}\log_5 8 + 13\log_5 1$

Solutions.

(a) $\ln 3 + 2\ln 4 = \ln 3 + \ln 4^2$ (prop. (vii))

$= \ln 3 + \ln 16$

$= \ln(3 \cdot 16)$ (prop. (v))

$= \ln 48$

(b) $\frac{1}{2}\log_5 49 - \frac{1}{3}\log_5 8 + 13\log_5 1 = \frac{1}{2}\log_5 49 - \frac{1}{3}\log_5 8$ (prop. (ii))

$= \log_5 (49)^{1/2} - \log_5 (8)^{1/3}$ (prop. (vii))

$= \log_5 7 - \log_5 2$

$= \log_5\left(\frac{7}{2}\right)$ (prop. (vi))

$= \log_5 (3.5)$

Example 11. Simplify: $e^{2\ln 3 - 3\ln 2}$

$e^{2\ln 3 - 3\ln 2} = e^{\ln 3^2 - \ln 2^3}$ (prop. (vii))

$= e^{\ln 9 - \ln 8}$

$= e^{\ln\left(\frac{9}{8}\right)}$ (prop. (vi))

$= \frac{9}{8}$ (prop. (iv))

Example 11. Given the approximate values

$$\log_{10} 2 \approx 0.3010 \qquad \text{and} \qquad \log_{10} 3 \approx 0.4771 \quad ,$$

find approximate values for the following: (a) $\log_{10} 24$ (b) $\log_{10} 5$ (c) $\log_{10} \sqrt[3]{4}$

Solutions.

(a)
$$\begin{aligned}
\log_{10} 24 &= \log_{10}(2^3 \cdot 3) \\
&= \log_{10} 2^3 + \log_{10} 3 && \text{(prop. (v))} \\
&= 3\log_{10} 2 + \log_{10} 3 && \text{(prop. (vii))} \\
&\approx 3(0.3010) + 0.4771 && \text{(given info.)} \\
&= 1.380
\end{aligned}$$

(b)
$$\begin{aligned}
\log_{10} 5 &= \log_{10} \frac{10}{2} \\
&= \log_{10} 10 - \log_{10} 2 && \text{(prop. (vi))} \\
&= 1 - \log_{10} 2 && \text{(prop. (i))} \\
&\approx 1 - 0.3010 && \text{(given info.)} \\
&= 0.699
\end{aligned}$$

[note: the relation $5 = 3 - 2$ is not useful here]

(c)
$$\begin{aligned}
\log_{10} \sqrt[3]{4} &= \log_{10} 2^{2/3} \\
&= \frac{2}{3} \log_{10} 2 && \text{(prop. (vii))} \\
&\approx \frac{2}{3}(0.3010) && \text{(given info.)} \\
&\doteq 0.2007
\end{aligned}$$

Example 12. Sketch the graph of $y = 1 + \log_2(x+1)$

$x = -1$ vertical asymptote

$x = 0 \rightarrow y = 1 + \log_2 1 = 1 + 0 = 1$

$(0,1)$ y-intercept

$y = 0 \rightarrow 1 + \log_2(x+1) = 0$

$\log_2(x+1) = -1$

$x + 1 = 2^{-1}$

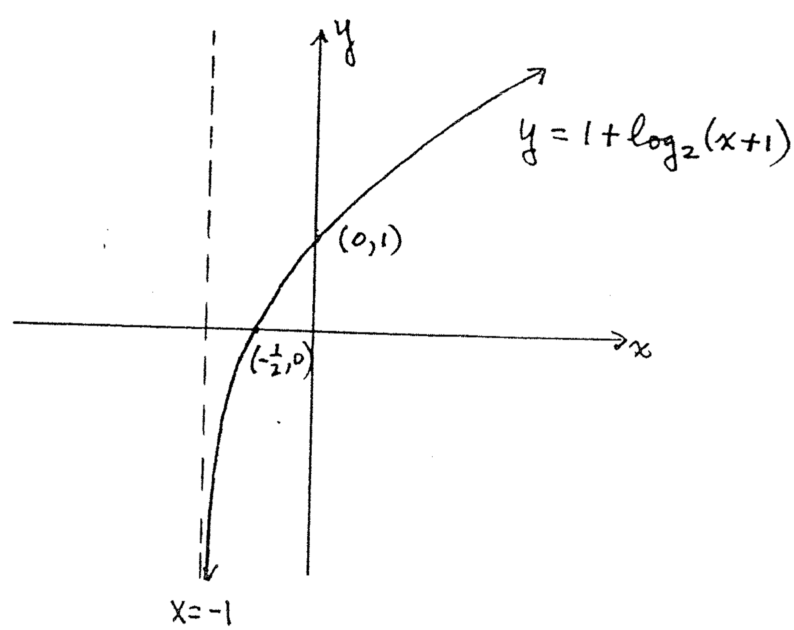
$x + 1 = \frac{1}{2}$

$x = -\frac{1}{2}$

$(-\frac{1}{2}, 0)$ x-intercept

domain: $x + 1 > 0$

$x > -1$



Exercises F

① Sketch the graphs of the following. Label intercepts and asymptotes. Give domain and range.
(a) $y = 2^x$ (b) $y = 2^{x^2}$ (c) $y = 2^{|x|}$ (d) $y = -2^x$ (e) $y = (\frac{3}{4})^x$
(f) $y = 3^x - 3$ (g) $y = 3^{-x+1}$

② Rewrite the given exponential statement in logarithmic form: (a) $4^{-1/2} = \frac{1}{2}$ (b) $9^0 = 1$
(c) $10^y = x$ (d) $e^y = 3$ (e) $\left(\frac{1}{64}\right)^{-1/2} = 8$ (f) $e^5 = x$ (g) $36^{-3/2} = \frac{1}{216}$

③ Rewrite the given logarithmic statement in exponential form: (a) $\log_3 81 = 4$
(b) $\log_{10} 10 = 1$ (c) $\log_x 5 = 2$ (d) $\log_2 x = y$ (e) $\ln x = 3$
(f) $\log_5 \frac{1}{25} = -2$ (g) $\ln 3x = -2$ (h) $\log_{16} 2 = \frac{1}{4}$ (i) $\ln e^2 = 2$

④ Find the value of the following: (a) $\log_{10} 0.000001$ (b) $\log_2 (2^2 + 2^2)$
(c) $\log_{64} \frac{1}{32}$ (d) $\log_{1/2} 16$ (e) $\ln \sqrt{e}$ (f) $\log_{5/2} \frac{8}{125}$ (g) $\ln (e^2 \cdot e^3)$
(h) $\ln (e^2)^3$ (i) $\log_4 \frac{1}{64}$ (j) $\log_7 \sqrt[3]{49}$ (k) $\log_{\sqrt{3}} 9$ (l) $\log_8 \frac{1}{4}$
(m) $\log_6 216$ (n) $\ln \frac{1}{\sqrt[3]{e^2}}$ (o) $\ln \left(\frac{e^{3/2}}{e^2 \sqrt{e}}\right)$

⑤ Solve for the unknown: (a) $\log_b 125 = 3$ (b) $\log_7 343 = x$
(c) $\log_2 \frac{1}{N} = 5$ (d) $2 \log_9 x = 1$ (e) $\log_3 \frac{1}{27} = x$ (f) $\ln x = 3$
(g) $\ln \sqrt{e^1} = x$ (h) $\log_{10} N = -2$ (i) $\log_5 25^c = 4$ (j) $\ln e^{2x} = -\frac{1}{2}$
(k) $\log_x 6 = -1$ (l) $\log_2 4^{-3} = x$ (m) $\ln e^4 = x^2$ (n) $\log_{10} \left(\frac{1}{1000}\right)^x = 1$

⑥ Use the approximate values $\log_{10} 4 \doteq 0.6021$ and $\log_{10} 5 \doteq 0.6990$ to compute:
(a) $\log_{10} 2$ (b) $\log_{10} 64$ (c) $\log_{10} \sqrt{40}$ (d) $\log_{10} \sqrt[3]{5}$ (e) $\log_{10} 0.8$

⑦ Use the approximate values $\ln 2 \doteq 0.6931$ and $\ln 3 \doteq 1.0986$ to compute:
(a) $\ln 6$ (b) $\ln 8$ (c) $\ln 2e^3$ (d) $\ln \frac{4}{3}$ (e) $\ln \frac{e}{9}$ (f) $\ln \frac{1}{27\sqrt{e}}$

⑧ Simplify and write as <u>one</u> logarithm: (a) $\log_{10} 2 + \log_{10} 5$

(b) $\log_{10}(x^4-4) - \log_{10}(x^2+2)$ (c) $3\ln 5 - \frac{1}{2}\ln 4 + \ln 8$

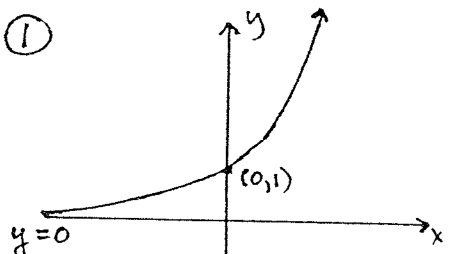
(d) $\log_2 5 + \log_2 5^2 + \log_2 5^3 - \log_2 5^6$ (e) $\ln(x^3+8) - \ln(x^2-2x+4) - 2\ln(x+2)$

⑨ Graph the following. Label intercepts and asymptotes. Give domain and range.

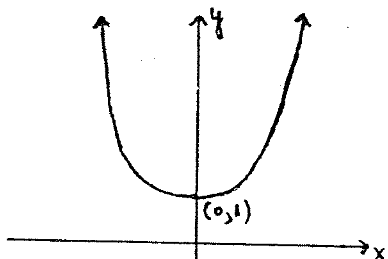
(a) $y = \log_2 x$ (b) $y = \log_2(x-3)$ (c) $y = \log_2 \sqrt{x}$ (d) $y = -1 + \log_2 x$

<u>Answers</u> (Exercises F)

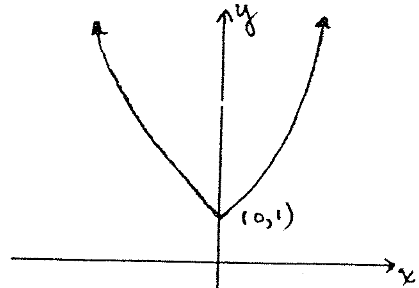
①



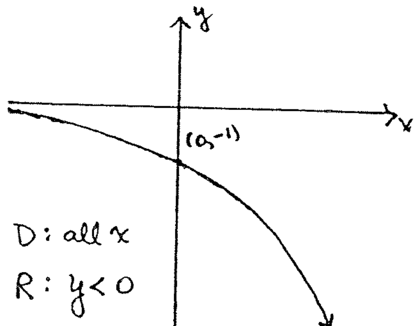
(a) D: all x R: $y > 0$



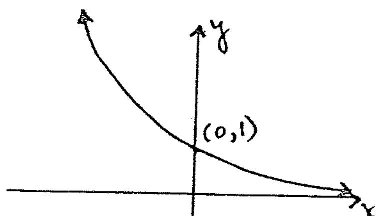
(b) D: all x R: $y \geq 1$



(c) D: all x R: $y \geq 1$

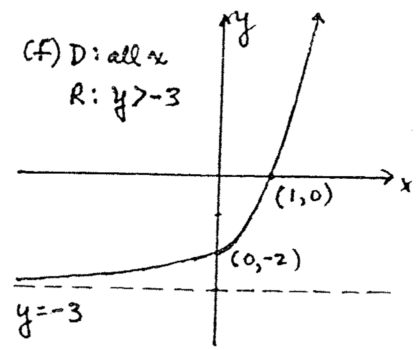


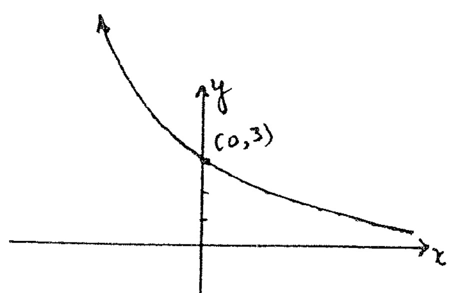
(d) D: all x R: $y < 0$



(e) D: all x R: $y > 0$

(f) D: all x R: $y > -3$





(g) D: all x R: $y > 0$

② (a) $\log_4 \frac{1}{2} = -\frac{1}{2}$ (b) $\log_9 1 = 0$ (c) $\log_{10} x = y$

(d) $\ln 3 = y$ (e) $\log_{1/64} 8 = -\frac{1}{2}$ (f) $\ln x = 5$

(g) $\log_{36} \frac{1}{216} = -\frac{3}{2}$

③ (a) $3^4 = 81$ (b) $10^1 = 10$ (c) $x^2 = 5$ (d) $2^y = x$

(e) $e^3 = x$ (f) $5^{-2} = \frac{1}{25}$ (g) $e^{-2} = 3x$ (h) $16^{1/4} = 2$

(i) $e^2 = e^2$

④ (a) -6 (b) 3 (c) $-\frac{5}{6}$ (d) -4 (e) $\frac{1}{2}$ (f) -3 (g) 5

(h) 6 (i) -3 (j) $\frac{2}{3}$ (k) 4 (l) $-\frac{2}{3}$ (m) 3 (n) $-\frac{2}{3}$ (o) -1

⑤ (a) 5 (b) 3 (c) $\frac{1}{32}$ (d) 3 (e) -3 (f) e^3 (g) $\frac{1}{2}$ (h) 0.01

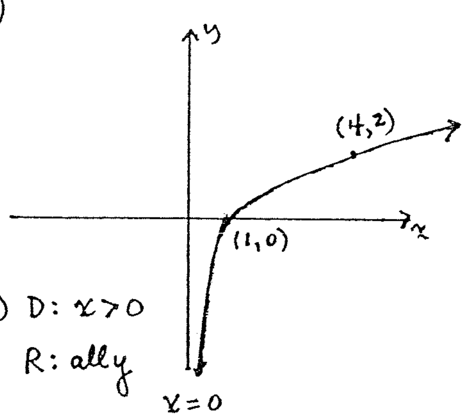
(i) 2 (j) $-\frac{1}{4}$ (k) $\frac{1}{6}$ (l) -6 (m) $2, -2$ (n) $-\frac{1}{3}$

⑥ (a) 0.301 (b) 1.806 (c) 0.801 (d) 0.233 (e) -0.0969

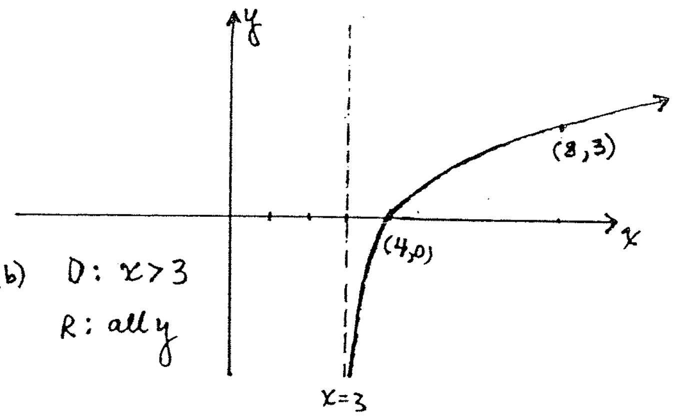
⑦ (a) 1.792 (b) 2.079 (c) 3.693 (d) 0.2877 (e) -1.197 (f) -3.796

⑧ (a) $\log_{10} 10 (= 1)$ (b) $\log_{10}(x^2 - 2)$ (c) $\ln 500$ (d) $\log_2 1 (= 0)$
(e) $-\ln(x+2)$

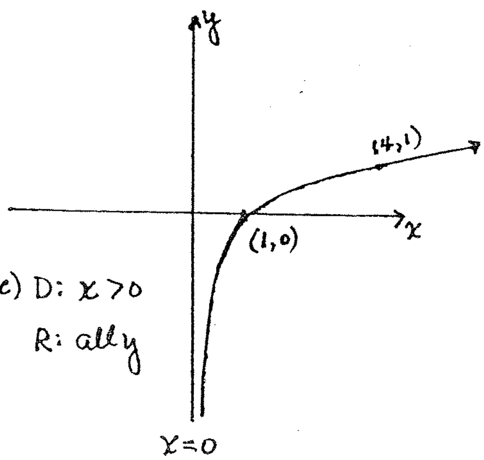
⑨



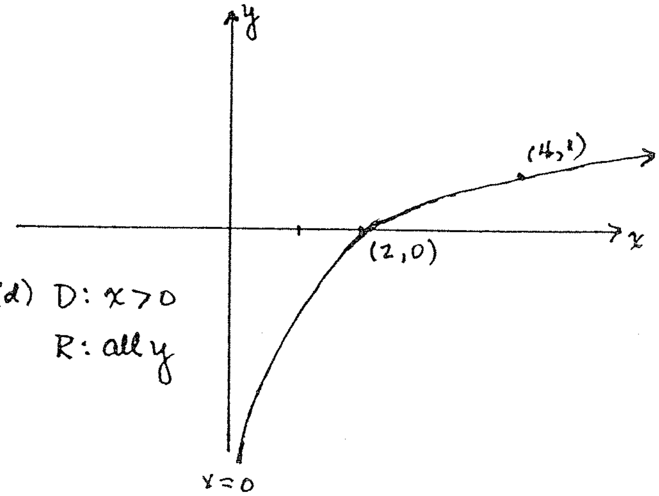
(a) D: $x > 0$
R: all y



(b) D: $x > 3$
R: all y



(c) D: $x > 0$
R: all y



(d) D: $x > 0$
R: all y

Section G. Exponential and Logarithmic Equations

I. Exponential Equations

An exponential equation is an equation in which the unknown appears in the exponent of one or more exponential expressions in the equation. For example,

$$7^{2x-1} = 1$$

$$4^{-x+1} = \frac{1}{32}$$

$$2^{x} + 2^{-x} = 2$$

are exponential equations. As with any equation, our objective is to find all values of the unknown which satisfy the equation.

A useful fact to remember when attempting to solve an exponential equation is that exponential functions are one-to-one so that whenever

$$b^{a} = b^{c},$$

we know that $a = c$.

Example 1. Solve for x: $7^{2x-1} = 1$

$$7^{2x-1} = 1$$

$$7^{2x-1} = 7^{0} \quad \text{(common base)}$$

$$2x - 1 = 0 \quad \text{(equate exponents)}$$

$$2x = 1$$

$$\boxed{x = \frac{1}{2}}$$

solution

Example 2. Solve for x: $e^{x^2-1} - 1 = 0$

$$e^{x^2-1} - 1 = 0$$
$$e^{x^2-1} = 1$$
$$e^{x^2-1} = e^0$$
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$\boxed{x = \pm 1}$$

Example 3. Solve for x: $4^{-x+1} = \frac{1}{32}$

$$4^{-x+1} = \frac{1}{32}$$
$$(2^2)^{-x+1} = 2^{-5} \qquad \text{(common base)}$$
$$2^{-2x+2} = 2^{-5}$$
$$-2x + 2 = -5$$
$$-2x = -7$$
$$\boxed{x = \frac{7}{2}}$$

Sometimes writing an exponential expression with one base is not as straightforward as it was in the preceding examples. In such cases we can replace the exponential expression containing the unknown with the equivalent logarithmic expression; for example

$$3^{5x-1} = 2 \longrightarrow \quad 5x-1 = \log_3 2 \text{ , from which}$$

we can solve for x in terms of $\log_3 2$.

Example 4. Solve for x: $2^{5x} = 3$

$$2^{5x} = 3$$

$$5x = \log_2 3 \qquad \text{(equivalent log. form)}$$

$$\boxed{x = \frac{1}{5}\log_2 3}$$

Example 5. Solve for x: $e^{2x+1} = 7$

$$e^{2x+1} = 7$$

$$2x+1 = \ln 7 \qquad \text{(equivalent log. form)}$$

$$2x = \ln 7 - 1$$

$$\boxed{x = \frac{\ln 7 - 1}{2}}$$

Sometimes algebraic manipulations are required to get an exponential equation into a form to which we can apply the principles discussed above.

Example 6. Solve for x: $27^x = \dfrac{9^{2x-1}}{3^{2x}}$

$$27^x = \frac{9^{2x-1}}{3^{2x}}$$

$$(3^3)^x = \frac{(3^2)^{2x-1}}{3^{2x}} \qquad \text{(common base)}$$

$$3^{3x} = \frac{3^{4x-2}}{3^{2x}}$$

$$3^{3x} = 3^{2x-2} \rightarrow 3x = 2x-2 \rightarrow \boxed{x = -2}$$

Example 7. Solve for x: $3^{2x} - 2(3^x) - 3 = 0$

$3^{2x} - 2(3^x) - 3 = 0$ [quadratic equ. in 3^x]

$(3^x - 3)(3^x + 1) = 0$ (factor - note $3^{2x} = (3^x)^2$)

$3^x - 3 = 0$ $3^x + 1 = 0$

$3^x = 3$ $\underbrace{3^x = -1}$

$\boxed{x = 1}$

no solution since $b^x > 0$ for all $b > 0$ and all x.

Example 8. Solve for x: $2^x + 2^{-x} = 2$

$2^x + 2^{-x} = 2$

$2^x\left[2^x + 2^{-x}\right] = 2^x(2)$ (mult. by 2^x)

$2^{2x} + 1 = 2(2^x)$

$2^{2x} - 2(2^x) + 1 = 0$

$(2^x - 1)^2 = 0$

$2^x = 1$

$\boxed{x = 0}$

II. Logarithmic Equations

A logarithmic equation is an equation in which the unknown appears in the argument of one or more logarithmic functions. For example,

$$\log_{10}(2x+50) = 2$$

$$\log_2 x + \log_2(x-2) = 3$$

$$\log_3(\log_2 x) = 1$$

are logarithmic equations.

<u>Example 9.</u> Solve for x: $\log_{10}(2x+50) = 2$

$$\log_{10}(2x+50) = 2$$

$$10^2 = 2x+50 \qquad \text{(equivalent exponential form)}$$

$$100 = 2x+50$$

$$2x = 50$$

$$\boxed{x = 25}$$

From this example we see that an equation of the form

$$\log_b(\text{expression involving } x) = \text{constant}$$

can be easily converted into another, simpler equation. Sometimes the properties of logarithms must be used to arrive at the form above, as the next two examples illustrate.

<u>Example 10.</u> $\log_2 x + \log_2(x-2) = 3$

$$\log_2 x + \log_2(x-2) = 3$$

$$\log_2[x(x-2)] = 3 \qquad \text{(combine log's with same base according to mult. rule)}$$

$$2^3 = x(x-2) \qquad \text{(equivalent exponential form)}$$

$$8 = x^2 - 2x$$
$$x^2 - 2x - 8 = 0$$

(quadratic equation — rearrange)

$$(x-4)(x+2) = 0$$

$$x = 4 \qquad x = -2$$

both answers should be substituted into the logarithmic functions appearing in the original equation. If an answer makes any argument 0 or a negative number then it cannot be a solution. [Remember that the domain of all logarithmic functions consists of positive numbers only.]

The only solution is $\boxed{x = 4}$

Example 11. Solve for x: $\log_3(7-x) = \log_3(1-x) + 1$

$$\log_3(7-x) = \log_3(1-x) + 1$$
$$\log_3(7-x) - \log_3(1-x) = 1$$
$$\log_3\left(\frac{7-x}{1-x}\right) = 1 \qquad \text{(div. property of logs)}$$
$$3^1 = \frac{7-x}{1-x}$$
$$7-x = 3(1-x)$$
$$7-x = 3-3x$$
$$2x = -4$$
$$\boxed{x = -2}$$

solution — check: arguments — all positive

In the next three examples, we must be careful not to use "nonproperties" of logarithms.

Example 12. Solve for x: $(\log_{10} x)^2 + \log_{10} x = 2$

$$(\log_{10} x)^2 + \log_{10} x = 2$$ [can't simplify $(\log_{10} x)^2$]

$$(\log_{10} x)^2 + \log_{10} x - 2 = 0$$ (quadratic in $\log_{10} x$)

$$(\log_{10} x + 2)(\log_{10} x - 1) = 0$$

$\log_{10} x = -2$ $\qquad$ $\log_{10} x = 1$

$x = 10^{-2}$ $\qquad$ $x = 10^1$

$\boxed{x = \frac{1}{100}}$ $\qquad$ $\boxed{x = 10}$

both are solutions - arguments checked

Example 13. Solve for x: $\dfrac{\log_3 16}{2 \log_3 x} = 2$

$$\frac{\log_3 16}{2 \log_3 x} = 2$$ (quotients of logs - can't simplify in general)

$$\log_3 16 = 4 \log_3 x$$

$$\log_3 16 - 4 \log_3 x = 0$$

$$\log_3 16 - \log_3 x^4 = 0$$

$$\log_3 \frac{16}{x^4} = 0$$

$$\frac{16}{x^4} = 3^0 = 1 \longrightarrow x^4 = 16 \rightarrow x = \pm 2; \text{ only}$$

$\boxed{x = 2}$ is solution

Example 14. Solve for x: $\log_3(\log_2 x) = 1$

$$\left.\begin{aligned} \log_3(\log_2 x) &= 1 \\ \log_2 x &= 3^1 = 3 \end{aligned}\right\} \text{two applications of exponential form}$$

$$x = 2^3$$

$$\boxed{x = 8}$$

Example 15. Solve for x: $9^{\log_3 x} = 4$

$$9^{\log_3 x} = 4$$

$$\left.\begin{aligned} (3^2)^{\log_3 x} &= 4 \\ 3^{2\log_3 x} &= 4 \\ 3^{\log_3 x^2} &= 4 \end{aligned}\right\} \text{rewrite left side to take advantage of relation } b^{\log_b x} = x$$

$$x^2 = 4$$

$x = \pm 2 \rightarrow$ only $\boxed{x = 2}$ is solution.

Example 16. Solve for x: $e^{-\ln x} = x$

$$e^{-\ln x} = x$$

$$e^{\ln x^{-1}} = x$$

$$x^{-1} = x$$

$$\frac{1}{x} = x$$

$x^2 = 1 \rightarrow x = \pm 1$ but $\boxed{x = 1}$ is only solution.

Exercises G

① Simplify: (a) $\log_{16}\frac{1}{\sqrt[5]{64}}$ (b) $e^{4\ln a+\frac{1}{2}\ln b}$ (c) $\frac{1}{\ln e^{-a}}$ (d) $\frac{1}{2}\log_4 16$

(e) $\frac{1}{2}\log_3 9-\frac{3}{4}\log_3 16+\frac{3}{2}\log_3 4$ (f) $e^{-\ln 5}$ (g) $e^{-\frac{1}{2}\ln\frac{1}{16}-\frac{2}{3}\ln 27}$

(h) $\log_3\sqrt[5]{\frac{1}{81}}$ (i) $\frac{1}{2}\log_3 64-2\log_3 2+\frac{1}{2}\log_3\frac{1}{4}$ (j) $\log_{1/2}64$

(k) $8^{\log_8 30}$ (l) $\log_{1/3}\sqrt[9]{(27)^2}$ (m) $2\log_5 4-\frac{1}{2}\log_5 64-\log_5 2$

(n) $\ln\left[e^{\ln 5-\frac{1}{2}\ln 25}\right]+\ln e^4-\ln\left[e^{\ln e}\right]$

② Solve the following exponential equations: (a) $5^{x-2}=1$ (b) $10^{-2x}=\frac{1}{10{,}000}$

(c) $2^x+2^{-x}=3$ (d) $2^{x^2}=8^{2x-3}$ (e) $\frac{1}{4}(10^{-2x})-25(10^x)=0$

(f) $4^{\log_2 x}=9$ (g) $e^{-\ln x}=5$ (h) $\left(\frac{1}{2}\right)^{2-x}=8(2^{x-1})^3$

(i) $6^{2x}-7(6^x)+6=0$ (j) $\frac{1}{3}=(2^{|x|-2}-1)^{-1}$

③ Solve the following logarithmic equations: (a) $\log_{10}\frac{1}{x^2}=2$

(b) $\log_3\sqrt{x^2+17}=2$ (c) $\log_2(\log_3 x)=2$ (d) $\log_4 x^2=(\log_4 x)^2$

(e) $\log_6 2x-\log_6(x+1)=0$ (f) $2\ln x=1$ (g) $\frac{\log_2 8^x}{\log_2\frac{1}{4}}=\frac{1}{2}$

(h) $\log_9\sqrt{10x+5}-\frac{1}{2}=\log_9\sqrt{x+1}$

④ Solve the following equations for x: (a) $\log_x 2=-\frac{1}{4}$

(b) $\log_5 x=-3$ (c) $\log_{1/2}8=x$ (d) $\log_x 4=\frac{2}{3}$ (e) $6^x=5$

(f) $\log_x b=3$ (g) $\ln x=-1$ (h) $2^{\ln x}=4$ (i) $\log_x 8=-\frac{3}{5}$

(j) $e^{\frac{1}{2}\ln(x+1)} = 3$ (k) $\ln x = e$ (l) $\log_2 2x + \log_2 (x+\frac{3}{2}) = 1$

(m) $\ln \sqrt{e^x} = -3$ (n) $2\log_2(x+2) - \log_2(x^2-4) = 3$ (o) $\log_4 \frac{2}{3x} = 3$

(p) $\log_x 8 = -\frac{3}{4}$ (q) $\frac{1}{2}\log_3(x^2-19) = 2$ (r) $2^{x^2+4x} = \frac{1}{8}$

Answers (Exercises 6)

1 (a) $-\frac{3}{10}$ (b) $a^4\sqrt{b}$ (c) $-\frac{1}{a}$ (d) 1 (e) 1 (f) $\frac{1}{5}$ (g) $\frac{4}{9}$ (h) $-\frac{4}{5}$

(i) 0 (j) -6 (k) 30 (l) $-\frac{2}{3}$ (m) 0 (n) 3

2 (a) 2 (b) 2 (c) $\log_2\left(\frac{3\pm\sqrt{5}}{2}\right)$ (d) 3 (e) $-\frac{2}{3}$ (f) 3 only (g) $\frac{1}{5}$

(h) -1 (i) 0, 1 (j) 4, -4

3 (a) $\pm\frac{1}{10}$ (b) ± 8 (c) 81 (d) 1, 16 (e) 1 (f) $\sqrt{e}$ (g) $-\frac{1}{3}$ (h) 4

4 (a) $\frac{1}{16}$ (b) $\frac{1}{125}$ (c) -3 (d) 8 (e) $\log_6 5$ (f) $\sqrt[3]{6}$ (g) $\frac{1}{e}$

(h) e^2 (i) $\frac{1}{32}$ (j) 8 (k) e^e (l) $\frac{1}{2}$ only (m) -6 (n) $\frac{18}{7}$ only

(o) $\frac{1}{96}$ (p) $\frac{1}{16}$ (q) ± 10 (r) $-3, -1$

Review Exercises – Part II (Sections F–G)

① Find the value: (a) $\log_{32}\sqrt[4]{8}$ (b) $\log_9\sqrt[4]{\frac{1}{27}}$ (c) $\log_{1/5}\sqrt[7]{(125)^3}$

(d) $\log_{1/4}\sqrt[6]{64}$ (e) $\ln\left(\frac{\sqrt{e^3}}{e}\right)$ (f) $e^{-\ln 3}$

② Simplify: (a) $\frac{2}{3}\log_{10}125 - \frac{1}{2}\log_{10}81 + \log_{10}\frac{18}{5}$

(b) $(e^{-\ln 3})(e^{4\ln\sqrt{3} - \frac{3}{2}\ln 16}) + (\ln e^{3/4})^2$ (c) $\log_7 48 + \log_7 21 - \log_7 9 - 2\log_7 4$

(d) $e^{\frac{1}{2}\ln\frac{1}{16} - \frac{2}{3}\ln 27} - \ln e^{5/4}$ (e) $\log_5\sqrt[3]{\frac{1}{625}} + \log_3 27 - \log_2 8 + \log_\pi 1$

(f) $8^{\log_2 3}$

③ Solve for x: (a) $1 + 2\log_3 x = \log_3(1-x) + \log_3(1+x)$

(b) $(\ln x)^2 = \ln x^2$ (c) $\log_x 36 = 2$ (d) $\log_{3/5} x = -3$ (e) $4^x + 1 = 6(4^{-x})$

(f) $\log_2(x-2) + \log_2(x-2) = 2$ (g) $\log_3(x^2 - 6x) = 3$ (h) $3^{x+1} = 27^{2x-3}$

(i) $\log_x 9 = -\frac{2}{3}$ (j) $e^{-\frac{1}{2}\ln x} = 4$ (k) $\frac{1}{2}\log_4(x+4) = 1$ (l) $\ln x^3 = \frac{1}{e}$

(m) $\left(\frac{1}{2}\right)^{2x+1} = 8$ (n) $\log_3(x-2) + \log_3(x+3) = \log_3 6$

(o) $[\log_3(x-2)]^2 + 3\log_3(x-2) = 0$ (p) $\ln x = \ln 1 + \ln 2 + \ln 3 + \ln 4$

(q) $\log_5(3x+4)^{1/2} - \log_5\sqrt{x} = 5^{\log_5 1}$ (r) $\log_6(x+3) + \log_6(x+4) = \log_6 6$

(s) $\frac{1}{6^x} = \sqrt[3]{36}$ (t) $4^{2x} - 5(4^x) + 4 = 0$ (u) $2^{x+1} = 4^{\frac{3}{2}x - 1}$

(v) $2^{x^2 - 5x + 6} = 1$ (w) $\log_2(x-7) + \log_2 x = \ln e^3$ (x) $e^{\ln 2x} - \ln e^{3x} = -3$

④ Sketch graphs of the following. Label intercepts and asymptotes. Give domain and range.

(a) $y = 2^{x+1} - 4$ (b) $y = \log_3(x-2) - 1$ (c) $y = 2^{-x} - 2$ (d) $y = \log_4(x+4)$

(e) $y = 3^{x-1}$ (f) $y = \log_2(x+1)$ (g) $y = \log_3(x-3)$

Answers (Review Exercises – Part II)

① (a) $\frac{3}{20}$ (b) $-\frac{3}{8}$ (c) $-\frac{9}{7}$ (d) $-\frac{1}{2}$ (e) $\frac{1}{2}$ (f) $\frac{1}{3}$

② (a) 1 (b) $\frac{39}{64}$ (c) 1 (d) $-\frac{11}{9}$ (e) $-\frac{4}{3}$ (f) 27

③ (a) $\frac{1}{2}$ only (b) $1, e^2$ (c) 6 only (d) $\frac{125}{27}$ (e) $\frac{1}{2}$ (f) 4 (g) $-3, 9$
(h) 2 (i) $\frac{1}{27}$ (j) $\frac{1}{16}$ (k) 12 (l) $e^{1/3e}$ (m) -2 (n) 3 only (o) $3, \frac{55}{27}$
(p) 24 (q) $\frac{2}{11}$ (r) -1 only (s) $-\frac{2}{3}$ (t) $0, 1$ (u) $\frac{3}{2}$ (v) $3, 2$
(w) 8 only (x) 3

④