Graphing Segment

Curve Sketching Checklist

Note: * 'd items are the only ones which apply to polynomial functions.

I. **Information from the function $f(x)$ itself:**

* A. **Intercepts**

B. **Asymptotes**

1. **Vertical Asymptotes:** $x = a$ is a VA if $\lim\limits_{x \to a} f(x) = \pm\infty$.
2. **Horizontal Asymptotes:** $y = b$ is a HA if $\lim\limits_{x \to \pm\infty} f(x) = b$.

C. Restrictions on the **domain** ("holes," excluded regions, etc.) and **discontinuities.**

II. **Information from the Derivative $f'(x)$:**

* A. **Horizontal Tangents:**
x-coordinates of points where tangent line is horizontal are found by solving $f'(x) = 0$.

B. **Vertical Tangents:**
x-coordinates are those values $x = c$ for which $f(c)$ exists, but $\lim\limits_{x \to c} f'(x) = \pm\infty$.

* C. **Intervals of Increase and Decrease:**

1. $f(x)$ is *increasing* on an interval if $f'(x) > 0$ for all x in that interval.
2. $f(x)$ is *decreasing* on an interval if $f'(x) < 0$ for all x in that interval.

* D. **Local Maxima and Minima** (1st derivative test):

1. Local *maximum* at $x = c$ if the sign of $f'(x)$ is as shown for x near c:

$$\begin{array}{rccc} \text{sign of } f'(x) & + & & - \\ & \longleftarrow & \vert & \longrightarrow \\ x\text{-values} & & c & \end{array}$$

and f is continuous at c .

2. Local *minimum* at $x = c$ if the sign of $f'(x)$ is as shown for x near c:

$$\begin{array}{rccc} \text{sign of } f'(x) & - & & + \\ & \longleftarrow & \vert & \longrightarrow \\ x\text{-values} & & c & \end{array}$$

and f is continuous at c .

III. **Information from the 2^{nd} Derivative $f''(x)$:**

* A. **Concavity:**

1. graph is concave *upward* on an interval if $f''(x) > 0$ for all x in that interval.
2. graph is concave *downward* on an interval if $f''(x) < 0$ for all x in that interval.

* B. **Points of Inflection:** *points* where the concavity changes, i.e.
$x = c$ is the x-coordinate of a point of inflection if f is continuous at c and the sign of $f''(x)$ for x near c is as shown below:

$$\begin{array}{rccc} \text{sign of } f''(x) & + & & - \\ & \longleftarrow & \vert & \longrightarrow \\ x\text{-values} & & c & \end{array} \qquad \text{or} \qquad \begin{array}{rccc} \text{sign of } f''(x) & - & & + \\ & \longleftarrow & \vert & \longrightarrow \\ x\text{-values} & & c & \end{array}$$

Part I
Polynomials

1.) $y = x^3 - 2x^2 + x - 2$

intercepts: $\underline{(0,-2)}$

$x^3 - 2x^2 + x - 2 = 0$

$x^2(x-2) + (x-2) = 0$

$(x^2+1)(x-2) = 0$

$x^2 = -1 \quad x = 2 \quad \underline{(2,0)}$

ϕ

inflection points:

$y'' = 6x - 4$

$6x - 4 = 0$

$x = \frac{2}{3}$

intervals: $\frac{2}{3}$

sign of y'' } $-$ $+$

$\underline{\left(\frac{2}{3}, -\frac{52}{27}\right)}$

concave up: $x > \frac{2}{3}$

concave down: $x < \frac{2}{3}$

horizontal tangents:

$y' = 3x^2 - 4x + 1$

$3x^2 - 4x + 1 = 0$

$(3x-1)(x-1) = 0$

$x = \frac{1}{3} \quad x = 1 \quad \underline{\left(\frac{1}{3}, -\frac{50}{27}\right)}$

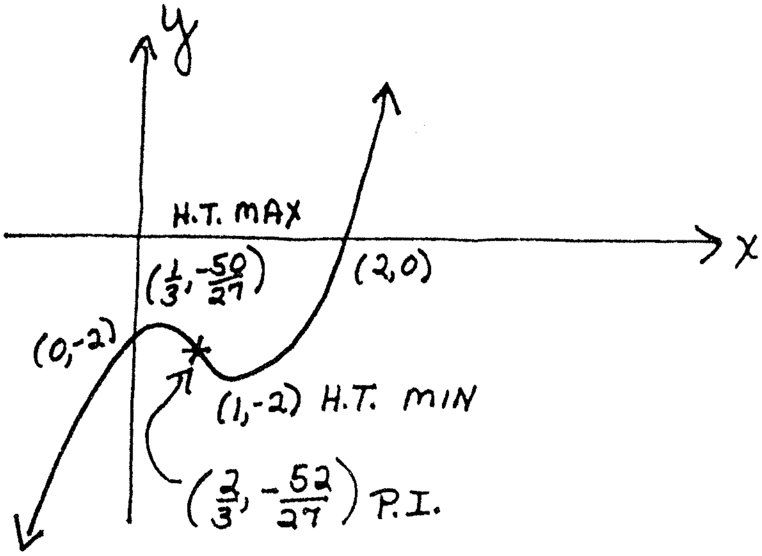
$\underline{(1,-2)}$

intervals: $\frac{1}{3}$ 1

$+$ MAX $-$ MIN $+$ {sign of y'

increasing: $x < \frac{1}{3} \quad x > 1$

decreasing: $\frac{1}{3} < x < 1$



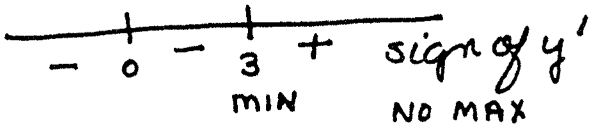
2.) $y = x^4 - 4x^3$

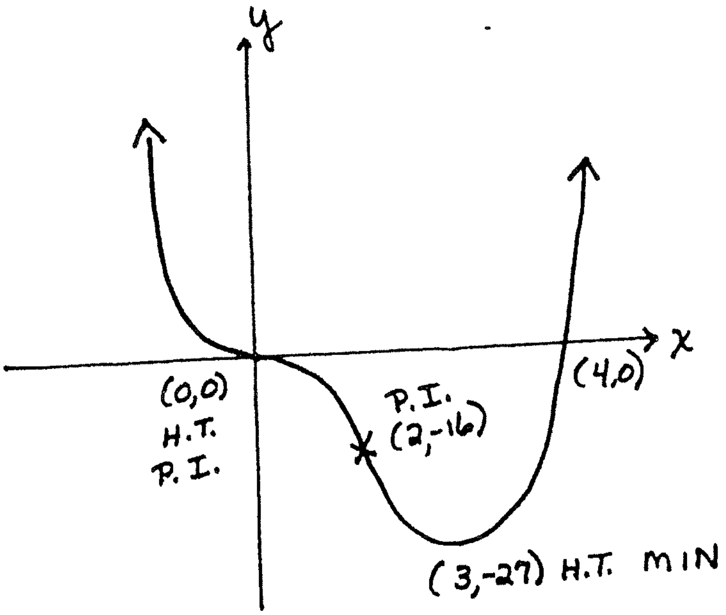
intercepts: $(0,0)$
$(4,0)$

horizontal tangents:

$y' = 4x^3 - 12x^2$ $(0,0)$

$x = 0,\ x = 3$ $(3,-27)$

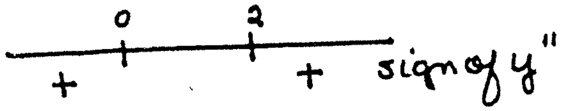




inflection points:

$y'' = 12x^2 - 24x$ $(0,0)$ P.I.

$x = 0$ $x = 2$ $(2,-16)$ P.I.



increasing: $x > 3$
decreasing: $x < 0,\ 0 < x < 3$
minimum: $(3,-27)$
concave up: $x < 0,\ x > 2$
concave down: $0 < x < 2$

Exercises Part I

Sketch the graph of each of the following:

① $y = x^3 - 4x^2 + 4x$ ② $y = 2x^3 - 9x^2 + 12x$

③* $y = x^5 - 5x^3 - 20x$ ④ $y = x^4 - 18x^2$ ⑤* $y = x^5 - 5x + 1$

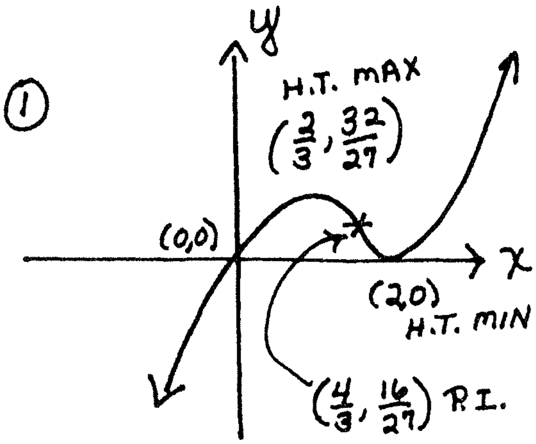
⑥ $y = x^4 - 8x^3 + 18x^2$ ⑦ $y = 2x^3 - 3x^2 - 12x + 18$

⑧* $y = x^3 - 3x^2 + 1$ ⑨ $y = (1-x)^2(1+x)^2$ ⑩ $y = \frac{1}{16}x^4 - 2x$

⑪* $y = x^3 + x^2 + 6x - 5$

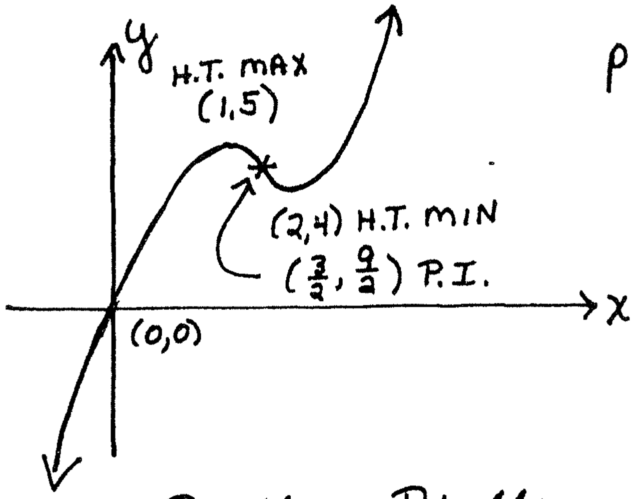
* do not attempt finding
x-intercepts for 3, 5, 8, 11

①



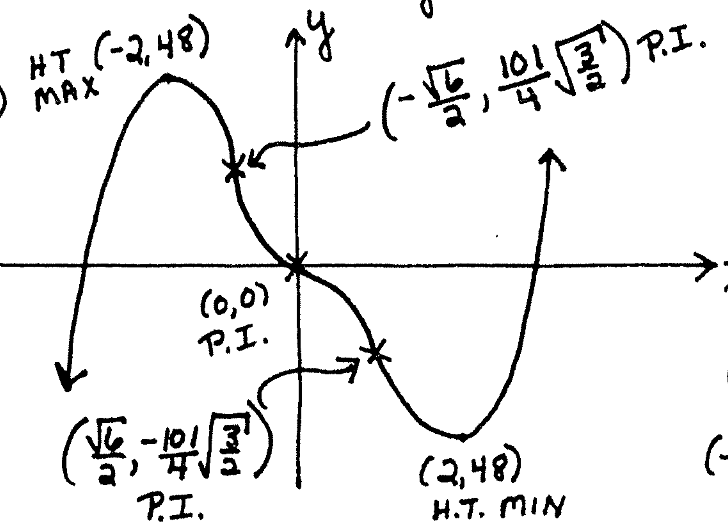
D: all x R: all y

②



D: all x R: all y

③

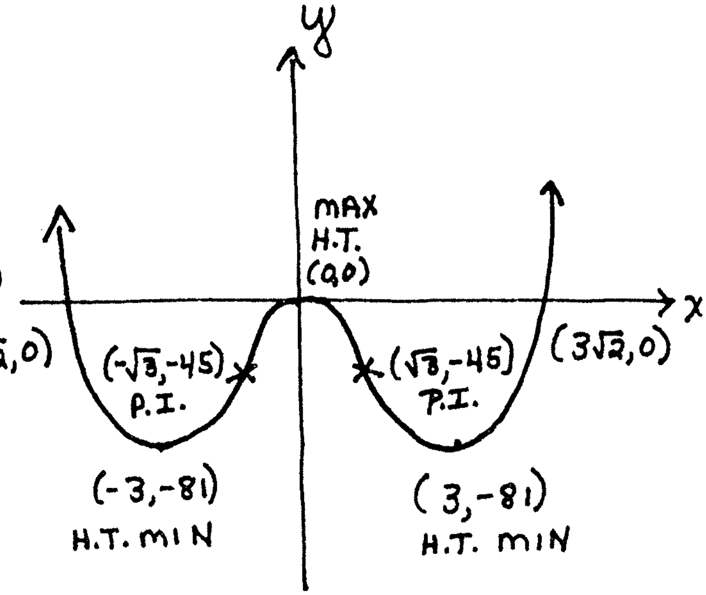


D: all x R: all y

concave down: $x < -\frac{\sqrt{6}}{2}, 0 < x < \frac{\sqrt{6}}{2}$

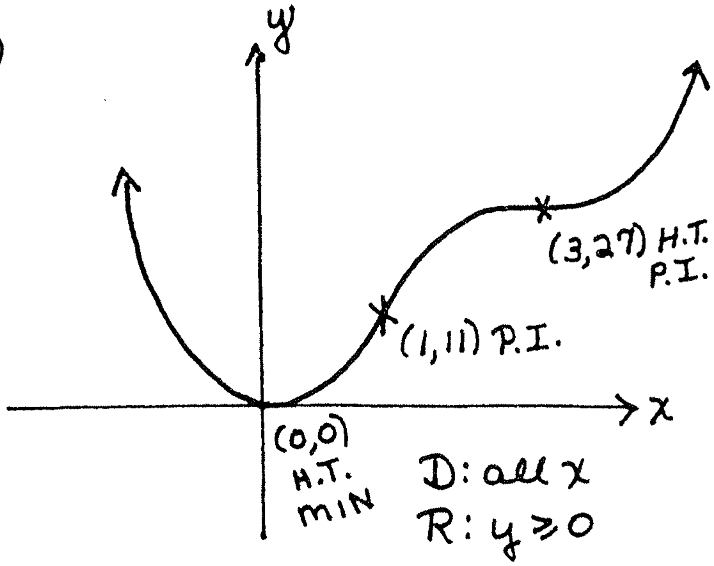
concave up: $-\frac{\sqrt{6}}{2} < x < 0, x > \frac{\sqrt{6}}{2}$

④



D: all x R: y ≥ -81

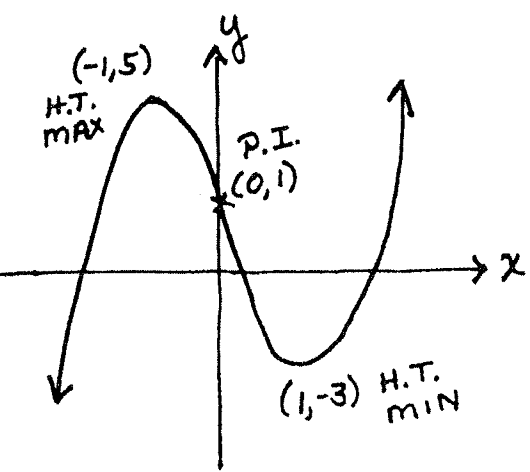
⑥



D: all x

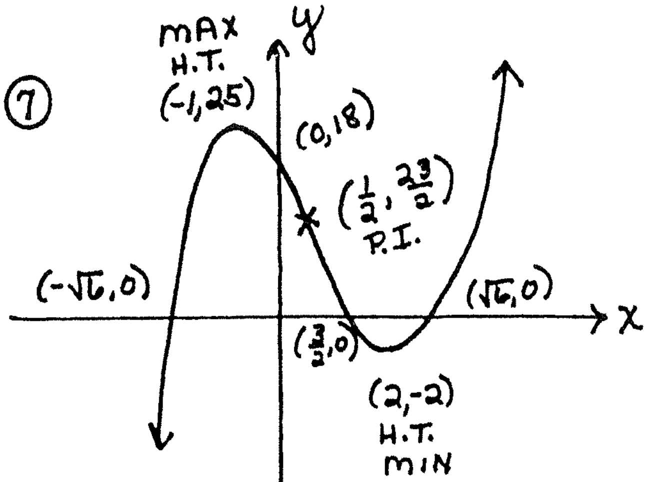
R: y ≥ 0

⑤



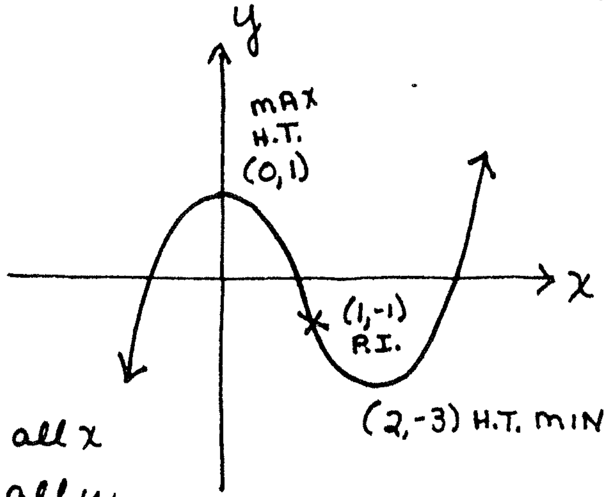
D: all x R: all y

⑦



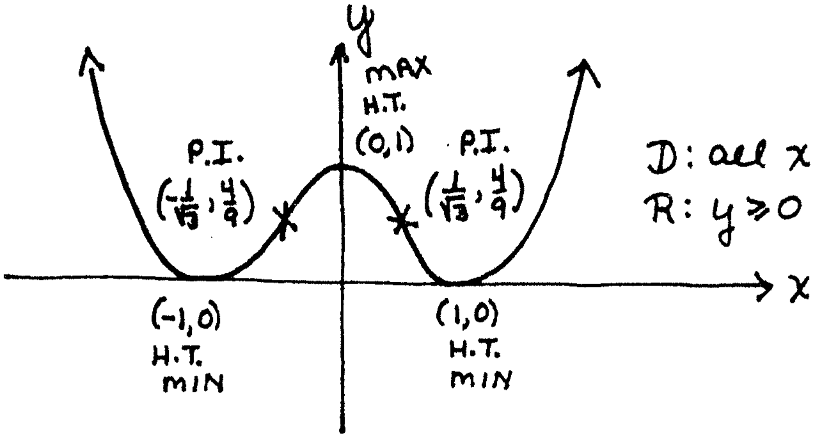
D: all x
R: all y

⑧



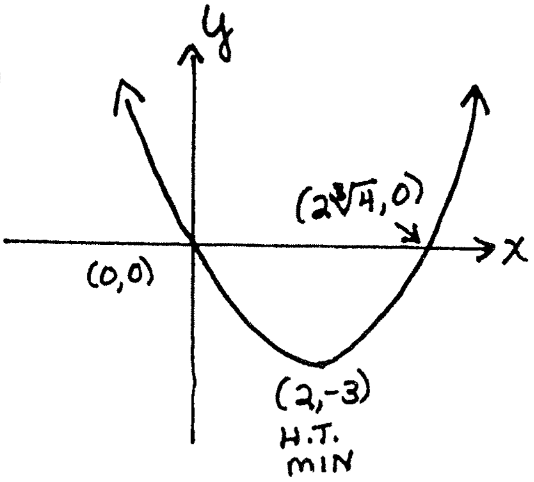
D: all x
R: all y

⑨



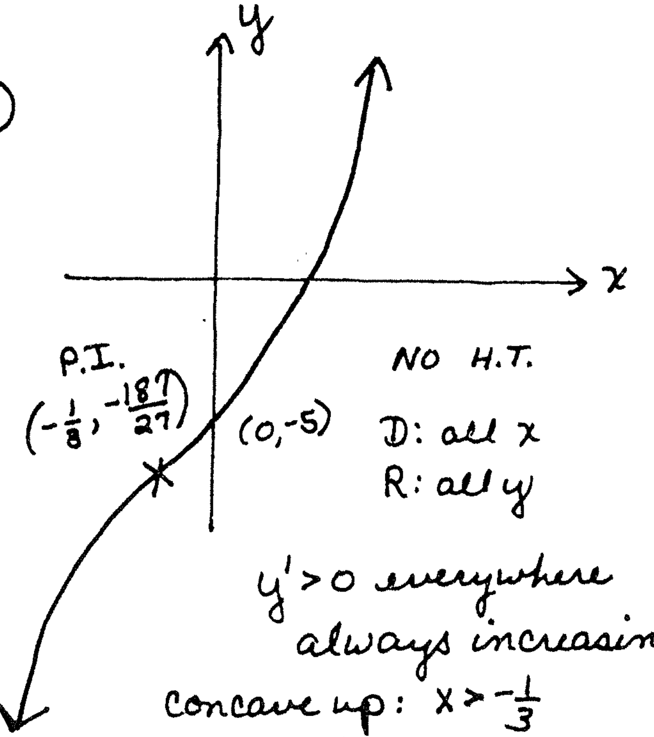
D: all x
R: $y \geq 0$

⑩



NO P.I.
D: all x
R: $y \geq -3$

⑪



NO H.T.
D: all x
R: all y

$y' > 0$ everywhere
always increasing
concave up: $x > -\frac{1}{3}$
concave down: $x < -\frac{1}{3}$

Part II

Infinite Limits, Limits at Infinity

Vertical asymptotes:

$x = a$ is a vertical asymptote if

$$\lim_{x\to a} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a} f(x) = -\infty$$

$$\lim_{x\to a^-} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty$$

$$\lim_{x\to a^+} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty$$

To find:

1.) Check where $f(x)$ is undefined.

2.) take right and left limits around these points.

Examples:

① $f(x) = \dfrac{1}{x^2-2x} = \dfrac{1}{x(x-2)}$ is undefined at $x = 0, 2$

$$\lim_{x\to 0^-} \frac{1}{\underset{\substack{\uparrow\\ \text{neg}}}{x}\underbrace{(x-2)}_{\text{neg}}} = +\infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{\underset{\substack{\uparrow\\ \text{pos}}}{x}\underbrace{(x-2)}_{\text{neg}}} = -\infty$$

$$\lim_{x\to 2^-} \frac{1}{\underset{\substack{\uparrow\\ \text{pos}}}{x}\underbrace{(x-2)}_{\text{neg}}} = -\infty \quad \text{and} \quad \lim_{x\to 2^+} \frac{1}{\underset{\substack{\uparrow\\ \text{pos}}}{x}\underbrace{(x-2)}_{\text{pos}}} = +\infty$$

$x = 0$ and $x = 2$ are vertical asymptotes

② $f(x) = \dfrac{x^2+7x+10}{x^2-x-6} = \dfrac{(x+5)(x+2)}{(x-3)(x+2)}$ is undefined at $x = 3, -2$

$$\lim_{x\to -2} \frac{(x+5)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = -\frac{3}{5}$$

There will be a "hole" in the graph at $(-2, -\frac{3}{5})$

$$\lim_{x\to 3^-} \frac{\overset{\text{pos}}{x+5}}{\underset{\text{neg}}{x-3}} = -\infty \quad \text{and} \quad \lim_{x\to 3^+} \frac{\overset{\text{pos}}{x+5}}{\underset{\text{pos}}{x-3}} = +\infty$$

$x = 3$ is a vertical asymptote

③ $f(x) = \dfrac{x}{(x+4)^2}$ is undefined at $x = -4$

$$\lim_{x\to -4^-} \frac{\overset{\text{neg}}{x}}{\underset{\text{pos}}{(x+4)^2}} = -\infty \quad \text{and} \quad \lim_{x\to -4^+} \frac{\overset{\text{neg}}{x}}{\underset{\text{pos}}{(x+4)^2}} = -\infty$$

$x = -4$ is a vertical asymptote

④ $f(x) = \dfrac{x^2}{(x+4)^2}$ is undefined at $x = -4$

$$\lim_{x\to -4^-} \frac{\overset{\text{pos}}{x^2}}{\underset{\text{pos}}{(x+4)^2}} = +\infty \qquad \lim_{x\to -4^+} \frac{\overset{\text{pos}}{x^2}}{\underset{\text{pos}}{(x+4)^2}} = +\infty$$

$x = -4$ is a vertical asymptote

Horizontal asymptotes:

$y=b$ is an horizontal asymptote if

$$\lim_{x\to+\infty} f(x)=b \quad \text{or} \quad \lim_{x\to-\infty} f(x)=b$$

To find:

1.) multiply $f(x)$ by $\left(\frac{1}{x^n}\right) \div \left(\frac{1}{x^n}\right)$ where n is the largest power in the denominator

2.) take $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$

Examples:

① $f(x)=\dfrac{1}{x^2-2x}$

$$\frac{1}{x^2-2x}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{\frac{1}{x^2}}{1-\frac{2}{x}}$$

$$\lim_{x\to+\infty}\frac{\cancelto{0}{\frac{1}{x^2}}}{1-\cancelto{0}{\frac{2}{x}}} = 0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{\cancelto{0}{\frac{1}{x^2}}}{1-\cancelto{0}{\frac{2}{x}}} = 0$$

$y=0$ is an horizontal asymptote

② $f(x)=\dfrac{3x^3-1}{x^3}$

$$\frac{3x^3-1}{x^3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{3-\frac{1}{x^3}}{1}$$

$$\lim_{x\to+\infty}\frac{3-\cancelto{0}{\frac{1}{x^3}}}{1} = 3 \quad \text{and} \quad \lim_{x\to-\infty}\frac{3-\cancelto{0}{\frac{1}{x^3}}}{1} = 3$$

$y=3$ is an horizontal asymptote

③ $f(x) = \dfrac{x^3-4}{x^2+8}$ $\dfrac{x^3-4}{x^2+8} \cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} = \dfrac{x - \frac{4}{x^2}}{1 + \frac{8}{x^2}}$

$$\lim_{x\to-\infty}\left[\frac{\overset{\text{(NOTE)}}{\overset{\Downarrow}{x}} - \cancelto{0}{\frac{4}{x^2}}}{1 + \cancelto{0}{\frac{8}{x^2}}}\right] = -\infty$$

$$\lim_{x\to+\infty}\left[\frac{\overset{\text{(NOTE)}}{\overset{\Downarrow}{x}} - \cancelto{0}{\frac{4}{x^2}}}{1 + \cancelto{0}{\frac{8}{x^2}}}\right] = +\infty$$

No horizontal asymptote; however, these limits will help with graphing.

Exercises Part II

Evaluate the following limits:

1.) $\displaystyle\lim_{x\to+\infty} \frac{x^2+7x+10}{x^2-x-6}$ 2.) $\displaystyle\lim_{x\to-\infty} \frac{x^2-2x+1}{x+1}$

3.) $\displaystyle\lim_{x\to+\infty} \frac{2x^2-3}{x^2+1}$ 4.) $\displaystyle\lim_{x\to-3^+} \frac{x^3}{x^2-9}$

5.) $\displaystyle\lim_{x\to3} \frac{\sqrt{x^2-5}-2}{x-3}$ 6.) $\displaystyle\lim_{x\to-2^+} \frac{x^3-3x}{x+2}$

7.) $\displaystyle\lim_{x\to-\infty} -\frac{3x^3+x^2}{x^3-5}$ 8.) $\displaystyle\lim_{x\to4^-} \frac{x^3-3x^2-4x}{x^3-4x^2+x-4}$

1.) 1 2.) $-\infty$ 3.) 2 4.) $+\infty$

5.) $\frac{3}{2}$ 6.) $-\infty$ 7.) -3 8.) $\frac{20}{17}$

Part III
Rational Functions

1.) $g(x) = \dfrac{x-1}{x^2+3}$ (Do not check concavity)

x-intercept: $\dfrac{x-1}{x^2+3} = 0 \longrightarrow x = 1$ $\underline{(1,0)}$

y-intercept: $g(0) = \dfrac{0-1}{0+3} \longrightarrow y = -\frac{1}{3}$ $\underline{(0,-\frac{1}{3})}$

no vertical asymptotes since $x^2+3>0$ always

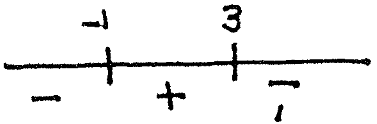
$$\lim_{x\to-\infty} \frac{x-1}{x^2+3}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty}\frac{\frac{1}{x}-\frac{1}{x^2}}{1+\frac{3}{x^2}} = \frac{0}{1} = 0$$

and $\displaystyle\lim_{x\to+\infty}\frac{\frac{1}{x}-\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$ therefore, $\underline{y=0 \text{ is an asymptote}}$

$$g'(x) = \frac{(x^2+3)(1)-(x-1)(2x)}{(x^2+3)^2} = \frac{-x^2+2x+3}{(x^2+3)^2}$$

$$g'(x)=0: \quad \left.\begin{aligned} -(x^2-2x-3)&=0\\ (x-3)(x+1)&=0 \end{aligned}\right\} \text{horizontal tangents at } x = 3,-1$$

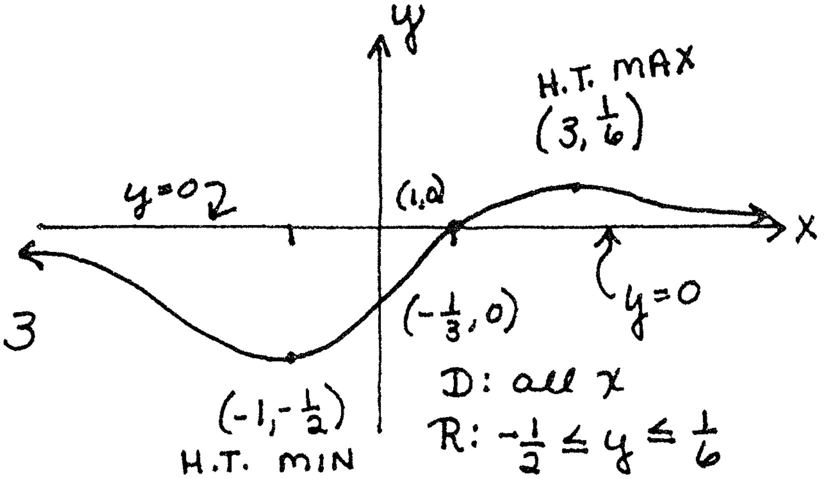
$g(3) = \frac{1}{6}$ and $g(-1) = -\frac{1}{2}$ $\underline{(3,\frac{1}{6})}$ $\underline{(-1,-\frac{1}{2})}$



sign of g'

increasing: $-1<x<3$

decreasing: $x<-1,\ x>3$



D: all x

R: $-\frac{1}{2} \le y \le \frac{1}{6}$

2.) $f(x) = \dfrac{x-5}{x^2-9}$ (Do not check concavity)

x-intercept: $(5,0)$ y-intercept: $(0, \frac{5}{9})$

asymptotes:

$$\lim_{x\to -3^-} \frac{x-5}{(x-3)(x+3)} = -\infty \qquad \lim_{x\to 3^-} \frac{x-5}{(x-3)(x+3)} = +\infty$$

$$\lim_{x\to -3^+} \frac{x-5}{(x-3)(x+3)} = +\infty \qquad \lim_{x\to 3^+} \frac{x-5}{(x-3)(x+3)} = -\infty$$

$\underline{x=-3}$ and $\underline{x=3}$ are asymptotes (both V.A.)

$$\lim_{x\to +\infty} \frac{\cancelto{0}{\frac{1}{x}} - \cancelto{0}{\frac{5}{x^2}}}{1 - \cancelto{0}{\frac{9}{x^2}}} = 0 \text{ and } \lim_{x\to -\infty} \frac{\cancelto{0}{\frac{1}{x}} - \cancelto{0}{\frac{5}{x^2}}}{1 - \cancelto{0}{\frac{9}{x^2}}} = 0$$

$\underline{y=0}$ is an asymptote (H.A.)

$$f'(x) = \frac{-x^2+10x-9}{(x^2-9)^2} \longrightarrow f'(x) = 0 = x^2-10x+9$$
$$0 = (x-1)(x-9)$$

horizontal tangents:

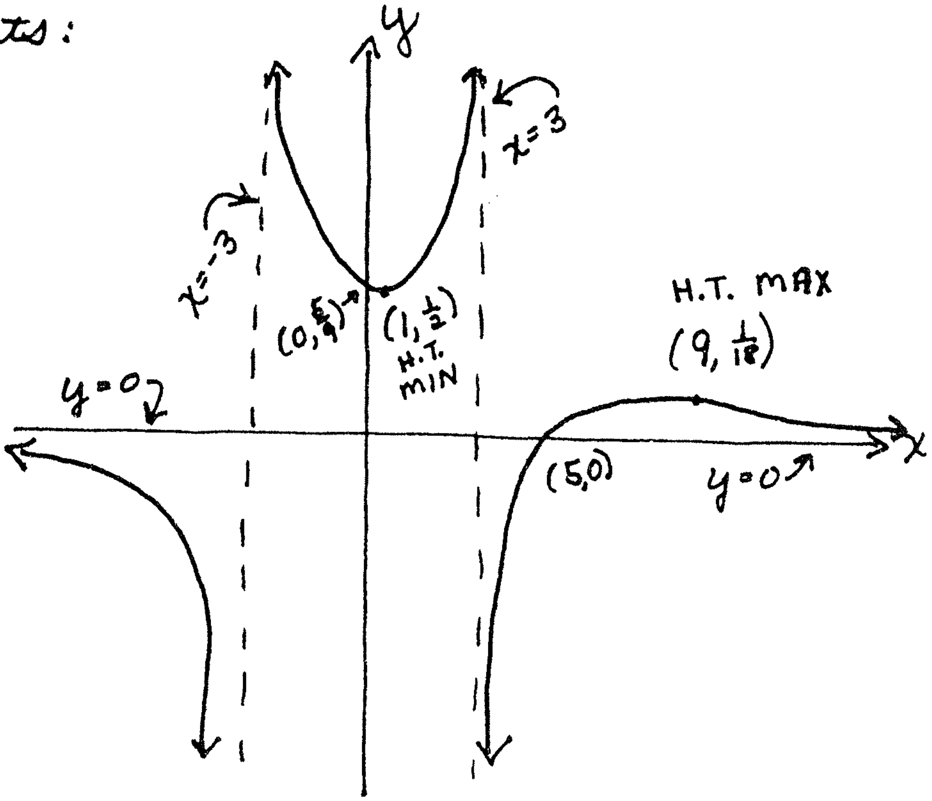
$(1, \frac{1}{2})$ $(9, \frac{1}{18})$

sign of f':

$-$		$-$		$+$		$+$		$-$
	-3		1		3		9	
	V.A.		H.T. MIN		V.A.		H.T. MAX	

increasing: $1<x<3$, $3<x<9$

decreasing: $x<-3$, $-3<x<1$, $x>9$



3.) $y = \dfrac{x^2}{(x-4)^2}$

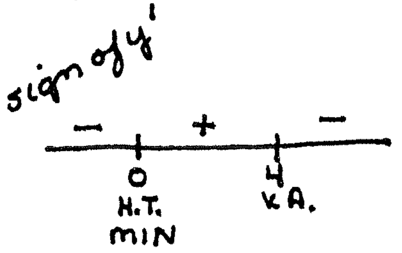
intercept: $\underline{(0,0)}$

$\lim\limits_{x \to 4^-} \dfrac{x^2}{(x-4)^2} = +\infty$; $\lim\limits_{x \to 4^+} \dfrac{x^2}{(x-4)^2} = +\infty$ $\underline{x=4}$ is a V.A.

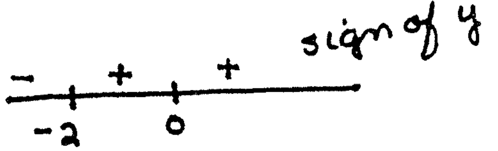
$\lim\limits_{x \to +\infty} \left[\dfrac{x^2}{x^2-8x+16} \cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} \right] = \lim\limits_{x \to +\infty} \dfrac{1}{1 - \cancel{\frac{8}{x}}^{\to 0} + \cancel{\frac{16}{x^2}}^{\to 0}} = 1$ $\underline{y=1}$ is H.A.

$\lim\limits_{x \to -\infty} \dfrac{x^2}{(x-4)^2} = 1$ also

$y' = \dfrac{-8x}{(x-4)^3} \longrightarrow x=0$ is H.T. $\underline{(0,0)}$

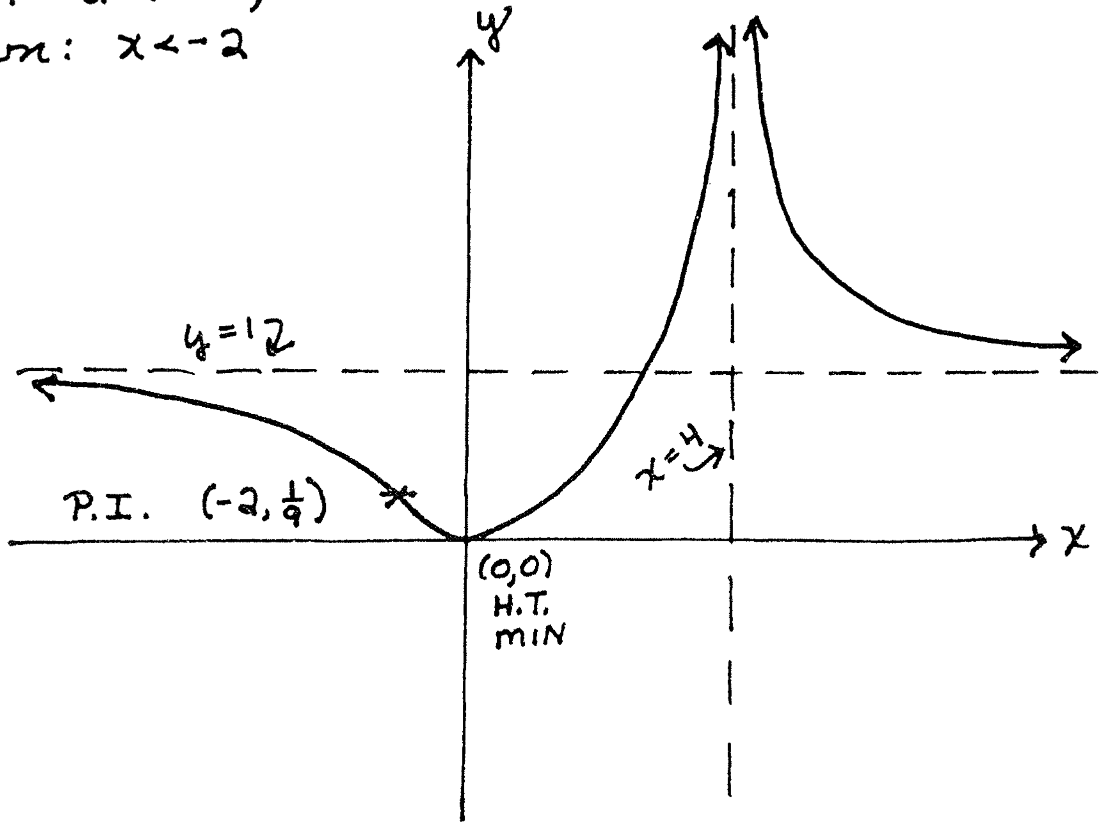


$y'' = \dfrac{16(2+x)}{(x-4)^4} \longrightarrow x = -2$



therefore, $\underline{(-2, \frac{1}{9})}$ is P.I.

increasing: $0 < x < 4$
decreasing: $x < 0,\ x > 4$
concave up: $-2 < x < 0,\ x > 0$
concave down: $x < -2$
D: $x \neq 4$
R: $y \geq 0$



4.) $y = \dfrac{x^3}{2(x^3+1)}$

intercept: $\underline{(0,0)}$

$\lim\limits_{x \to -1^-} \dfrac{x^3}{2(x^3+1)} = +\infty$ $\lim\limits_{x \to -1^+} \dfrac{x^3}{2(x^3+1)} = -\infty$ $\underline{x=-1}$ is V.A.

$\lim\limits_{x \to +\infty} \dfrac{x^3}{2x^3+2} \cdot \dfrac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim\limits_{x \to +\infty} \dfrac{1}{2+\cancel{\frac{2}{x^3}}^{\to 0}} = \dfrac{1}{2}$ $\underline{y=\frac{1}{2}}$ is H.A.

$\lim\limits_{x \to -\infty} \dfrac{1}{2+\frac{2}{x^3}} = \dfrac{1}{2}$ also

$y' = \dfrac{3x^2}{2(x^3+1)^2}$ $3x^2 = 0 \rightarrow \overset{\text{H.T.}}{\underline{(0,0)}}$

[no max or min]

$+ \quad \underset{\substack{-1 \\ \text{V.A}}}{|} \quad + \quad \underset{\substack{0 \\ \text{H.T.}}}{|} \quad +$ sign of y'

$y'' = \dfrac{3x(1-2x^3)}{(x^3+1)^3} \rightarrow x = 0, \sqrt[3]{\frac{1}{2}}$

$+ \quad \underset{\substack{-1 \\ \text{V.A.}}}{|} \quad - \quad \underset{0}{|} \quad + \quad \underset{\sqrt[3]{\frac{1}{2}}}{|} \quad -$ sign of y''

P.I. $\underline{(0,0)}$ and $\underline{\left(\sqrt[3]{\frac{1}{2}}, \frac{1}{6}\right)}$

increasing: $x \neq -1, 0$

concave up: $x < -1$

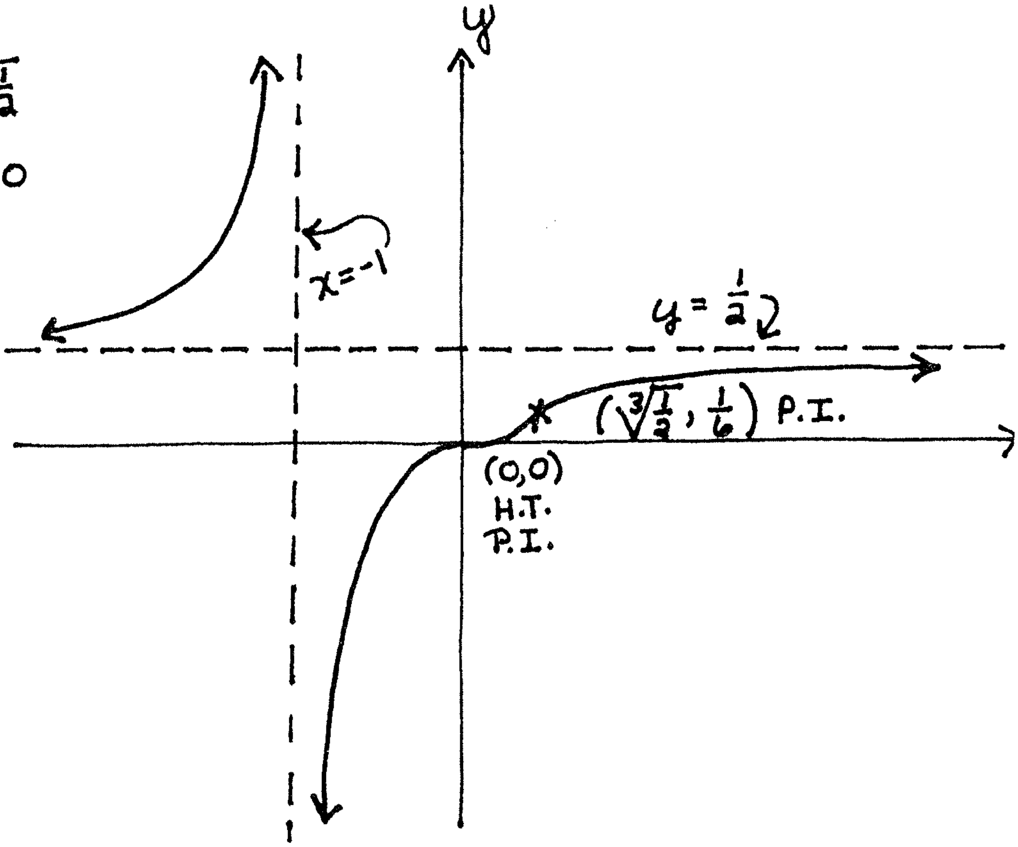
$0 < x < \sqrt[3]{\frac{1}{2}}$

concave down: $-1 < x < 0$

$x > \sqrt[3]{\frac{1}{2}}$

D: $x \neq -1$

R: $y \neq \frac{1}{2}$



5.) $f(x) = \dfrac{x^2-4}{x^2-x-6}$

$f(x) = \dfrac{x-2}{x-3}$, $x \neq -2$ is an equivalent form

$\lim\limits_{x \to -2} \dfrac{x-2}{x-3} = \dfrac{4}{5}$ and $x=-2$ is not in the domain of $f(x)$; therefore, $\underline{(-2, \frac{4}{5})}$ is a "hole" in the graph.

intercepts: $(2,0)$ and $(0, \frac{2}{3})$

$\lim\limits_{x \to 3^-} \dfrac{x-2}{x-3} = -\infty$ and $\lim\limits_{x \to 3^+} \dfrac{x-2}{x-3} = +\infty$ $\underline{x=3}$ is V.A.

$\lim\limits_{x \to +\infty} \dfrac{1-\frac{2}{x}}{1-\frac{3}{x}} = 1$ and $\lim\limits_{x \to -\infty} \dfrac{1-\frac{2}{x}}{1-\frac{3}{x}} = 1$ $\underline{y=1}$ is H.A.

$f'(x) = \dfrac{-1}{(x-3)^2} \rightarrow$ no horizontal tangents

$f'(x) < 0$ everywhere in domain

decreasing: $(-\infty,-2)$ $(-2,3)$ $(3,+\infty)$ [or $x \neq -2, 3$]

$f''(x) = \dfrac{2}{(x-3)^3} \rightarrow$ no inflection points

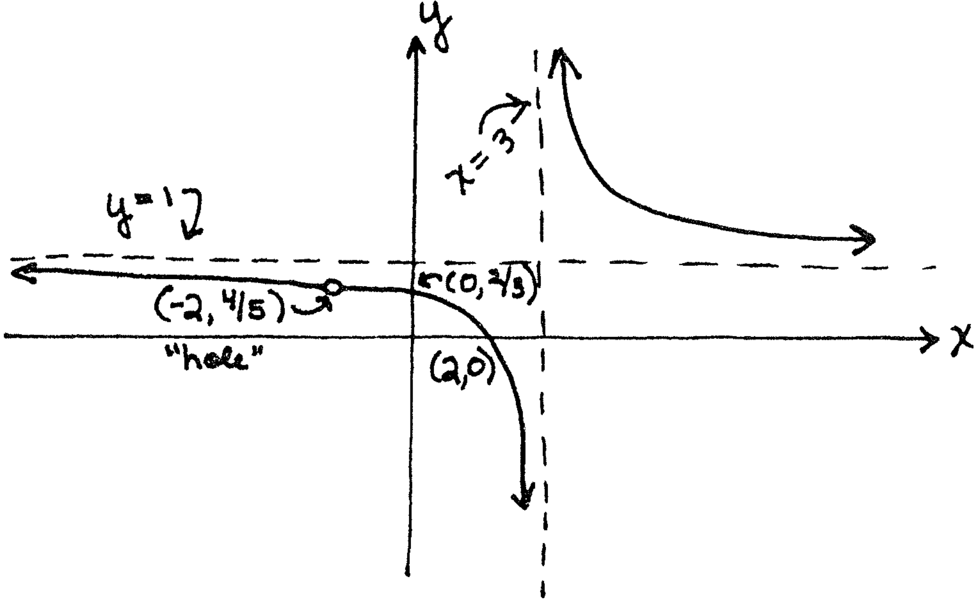
$\overset{-}{}\ \underset{-2}{|}\ \overset{-}{}\ \underset{3}{|}\ \overset{+}{}$ sign of $f''(x)$

concave down: $x < -2$
$-2 < x < 3$

concave up: $x > 3$

D: $x \neq -2, 3$

R: $y \neq \frac{4}{5}, 1$



6.) $f(x) = x^2 - \frac{1}{x} = \frac{x^3-1}{x}$

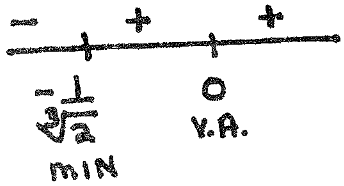
D: $x \neq 0 \rightarrow$ no y-intercept

x-intercept: $\underline{(1,0)}$

$\lim_{x\to 0^+}\left(x^2-\frac{1}{x}\right) = -\infty$; $\lim_{x\to 0^-}\left(x^2-\frac{1}{x}\right) = +\infty$ $x=0$ is V.A.

$\lim_{x\to +\infty}\left(x^2-\cancel{\frac{1}{x}}^{\to 0}\right) = +\infty$; $\lim_{x\to -\infty}\left(x^2-\cancel{\frac{1}{x}}^{\to 0}\right) = +\infty$ no H.A.

$f'(x) = 2x + x^{-2} = \frac{2x^3+1}{x^2}$ $x = -\frac{1}{\sqrt[3]{2}}$ is H.T. $\underline{\left(-\frac{1}{\sqrt[3]{2}}, \frac{3\sqrt[3]{2}}{2}\right)}$

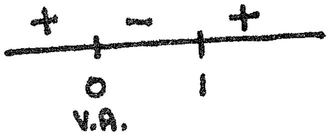


sign of f'

decreasing: $x < -\frac{1}{\sqrt[3]{2}}$

increasing: $-\frac{1}{\sqrt[3]{2}} < x < 0$, $x > 0$

$f''(x) = \frac{2x^3-2}{x^3} \longrightarrow x = 1$



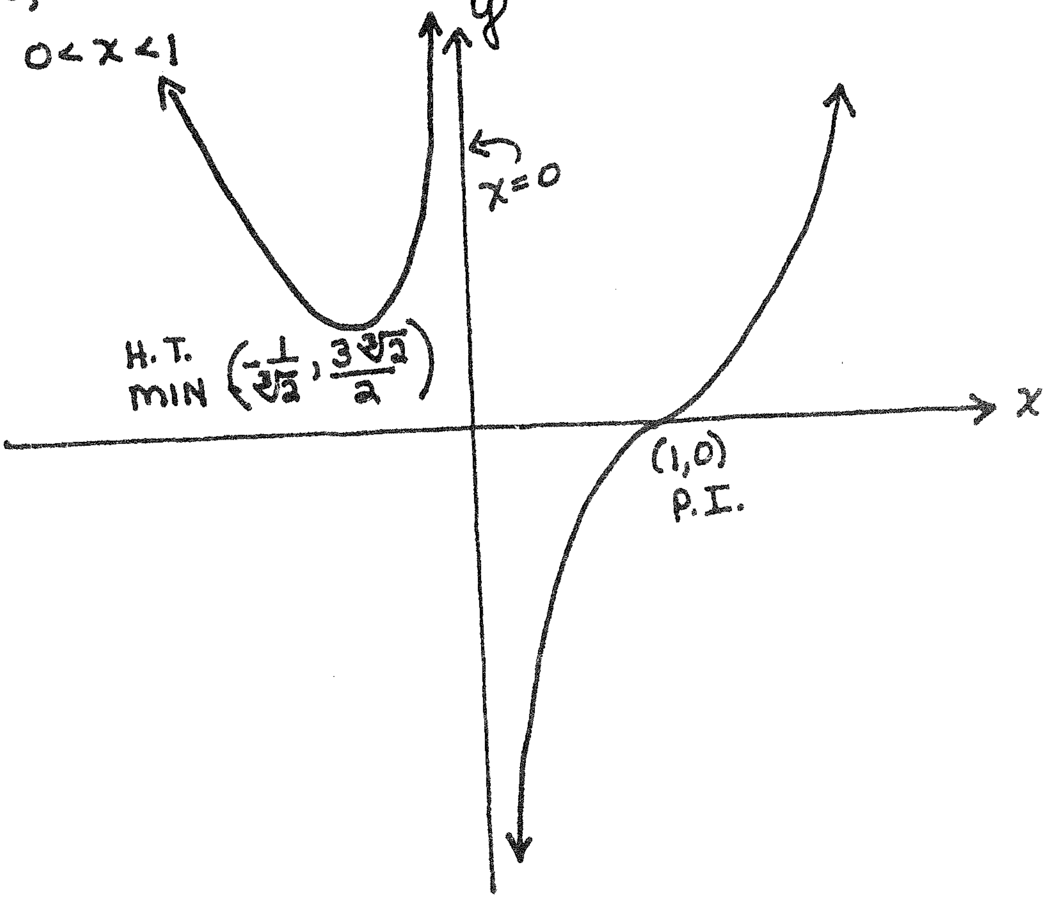
$\underline{(1,0)}$ P.I.

concave up: $x<0, x>1$

concave down: $0<x<1$

D: $x \neq 0$

R: all y



Exercises Part III

Sketch the graphs of the following. Include all important information.

Part A

1.) $y = \dfrac{x^2-9}{x^2-1}$ 2.) $y = \dfrac{2-x}{x^2+4x-12}$ 3.) $y = \dfrac{x+2}{x^2}$

4.) $y = \dfrac{-2x^2+32}{x^2+16}$ 5.) $f(x) = \dfrac{x-3}{x^2-9}$ 6.) $f(x) = \dfrac{x}{(x+1)^2}$

Part B

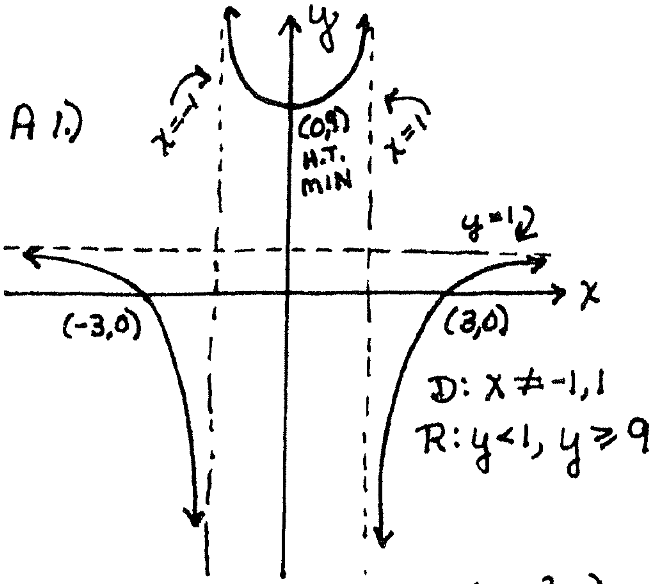
1.) $y = \dfrac{-2x}{x^2+1}$ 2.) $y = \dfrac{x+1}{x^2-3x}$ (omit concavity)

3.) $f(x) = \dfrac{4x^2}{2x^2+1}$ 4.) $y = \dfrac{2x+10}{9-x^2}$ (omit concavity)

5.) $y = \dfrac{x^2-3}{x^3}$ 6.) $f(x) = \dfrac{2x^2+5}{x^2+3}$

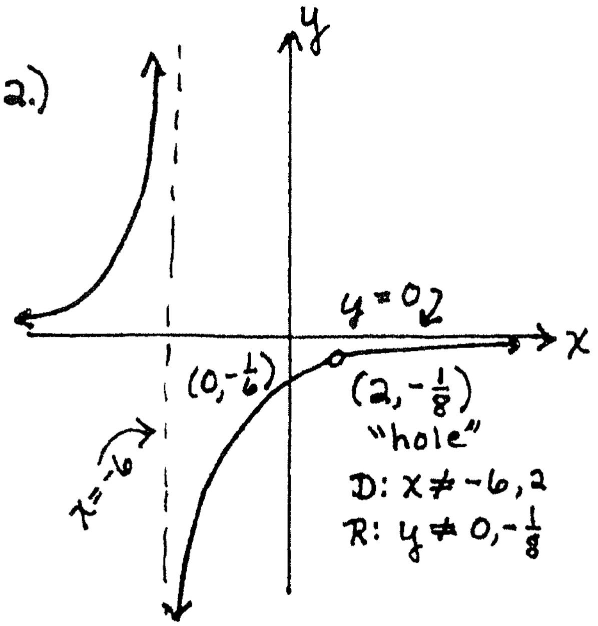
7.) $y = \dfrac{3x^2}{2+x-x^2}$ (omit concavity)

A 1.)

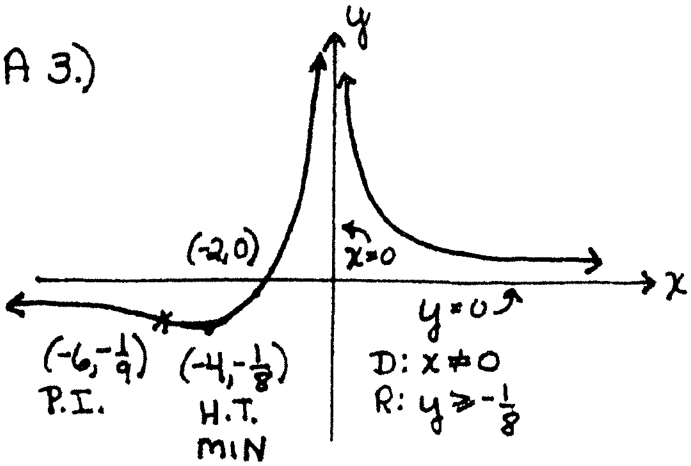


$$y' = \frac{16x}{(x^2-1)^2} \qquad y'' = \frac{-16(3x^2+1)}{(x^2-1)^3}$$

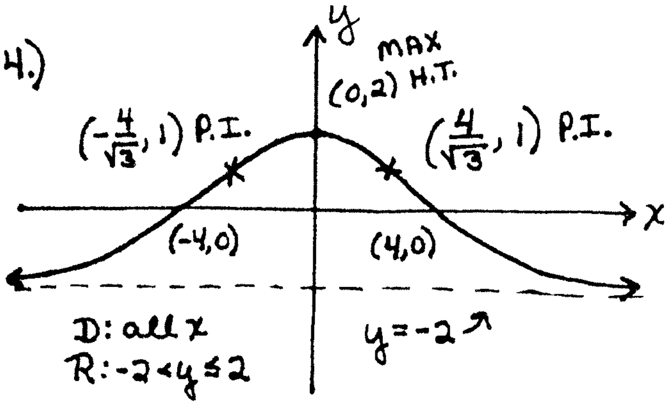
A 2.)



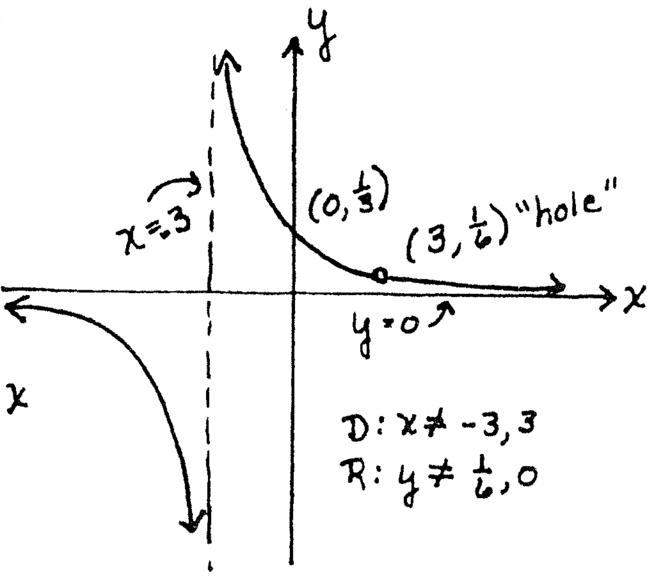
A 3.)



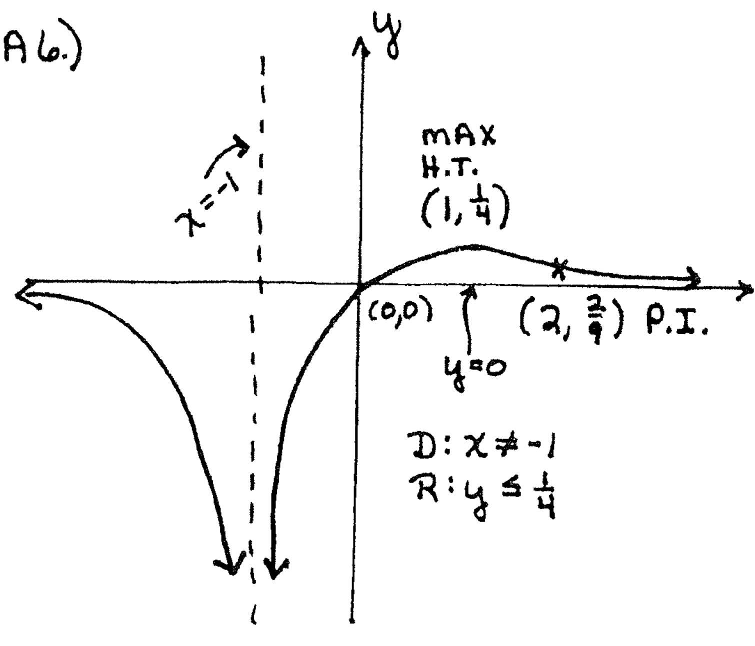
A 4.)



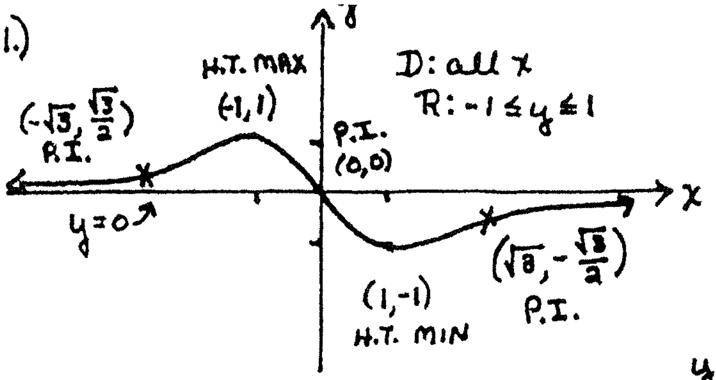
A 5.)



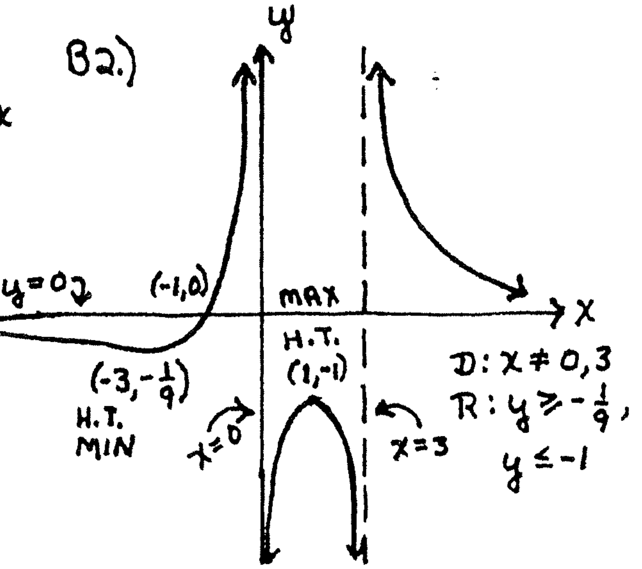
A 6.)



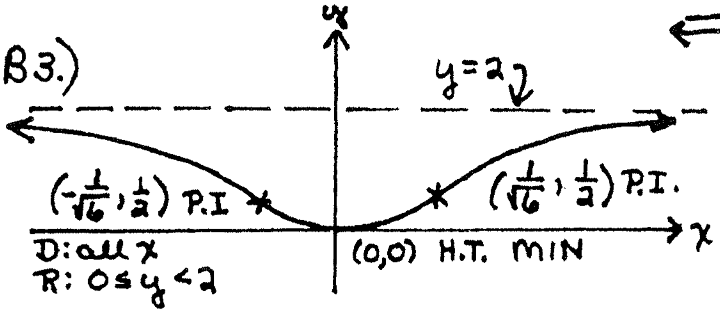
B1.)



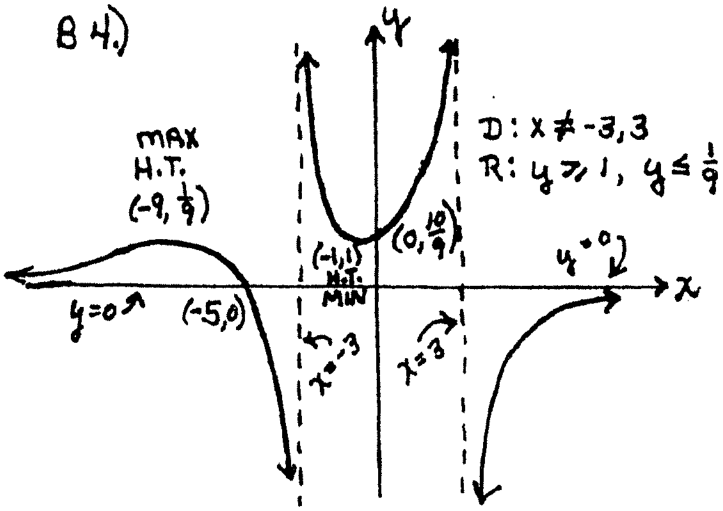
B2.)



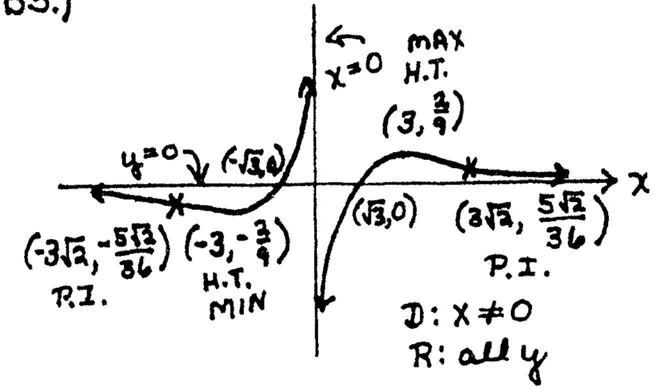
B3.)



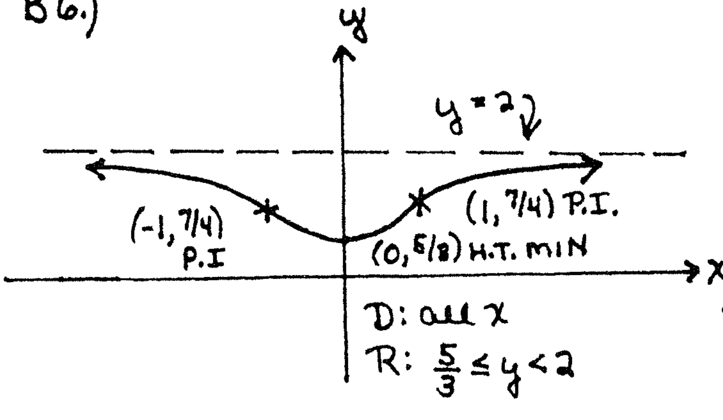
B4.)



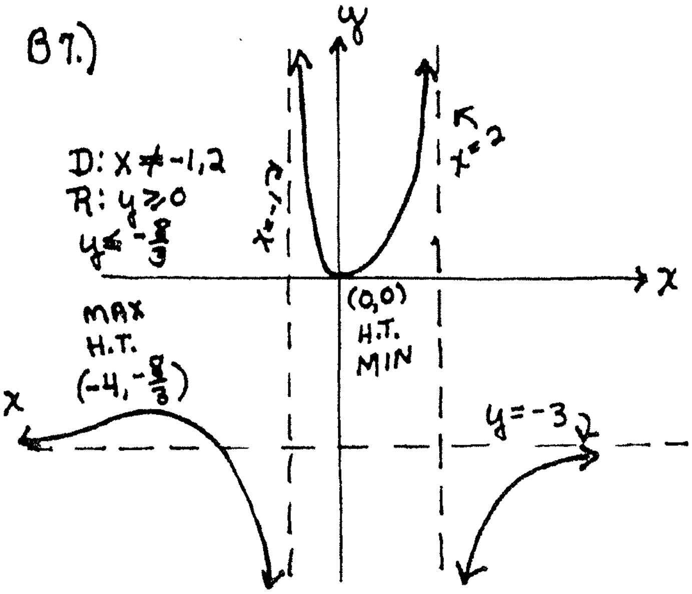
B5.)



B6.)



B7.)



Part IV

Vertical Tangents

1.) $f(x) = x^{2/3}(x-2)^2$ [Do not check concavity.]

intercepts: $\underline{(0,0)}$ $\underline{(2,0)}$

$$f'(x) = x^{2/3}\left[2(x-2)\right] + \frac{2}{3}x^{-\frac{1}{3}}(x-2)^2 = 2x^{2/3}(x-2) + \frac{2(x-2)^2}{3x^{1/3}}$$

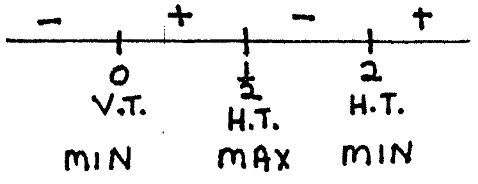
$$f'(x) = \frac{6x(x-2)+2(x-2)^2}{3x^{1/3}} = \frac{4(x-2)(2x-1)}{3x^{1/3}}$$

$f'(0)$ DNE *since* $f'(x)$ *is undefined at* $x=0$ *(but* $x=0$ *is in the domain of* $f(x)$*); therefore,* $(0,0)$ *is a* V.T.

$$\left.\begin{array}{l} 4(x-2)(2x-1)=0 \\ \quad x=2 \quad x=\frac{1}{2} \end{array}\right\} \quad \underline{(2,0)} \text{ and } \underline{\left(\frac{1}{2}, \frac{9\sqrt[3]{2}}{8}\right)} \text{ are H.T.}$$

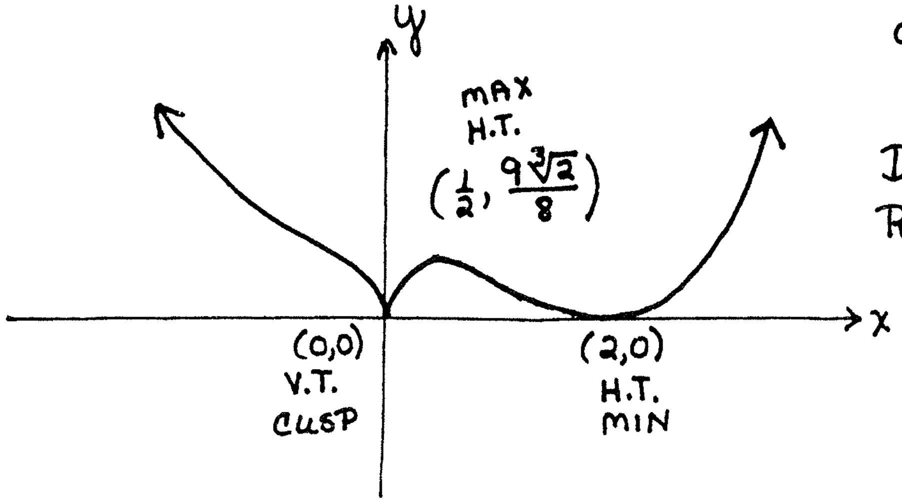
$$\left[\text{NOTE: } f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^{2/3}\left(\tfrac{1}{2}-2\right)^2 = \frac{1}{\sqrt[3]{4}}\cdot\frac{9}{4} = \frac{9}{4\sqrt[3]{4}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{9\sqrt[3]{2}}{4\cdot 2}\right]$$

sign of $f'(x)$:



increasing: $0 < x < \frac{1}{2}$, $x > 2$

decreasing: $x < 0$, $\frac{1}{2} < x < 2$



D: all x

R: $y \geq 0$

2.) $y = x^{2/3}(x+5)^{1/3}$ [Do not check concavity.]

intercepts: (0,0) (-5,0)

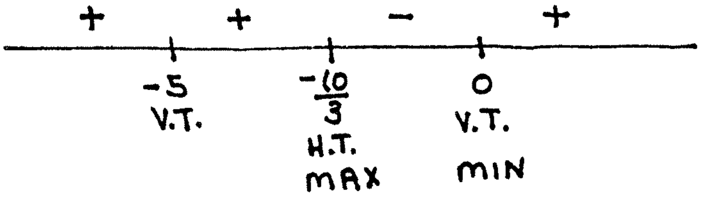
$$y' = x^{2/3}\left[\tfrac{1}{3}(x+5)^{-\frac{2}{3}}\right] + (x+5)^{1/3}\left[\tfrac{2}{3}x^{-\frac{1}{3}}\right] = \frac{x^{2/3}}{3(x+5)^{2/3}} + \frac{2(x+5)^{1/3}}{3x^{1/3}}$$

$$y' = \frac{3x+10}{3x^{1/3}(x+5)^{2/3}} \Rightarrow \text{vertical tangents at } x = 0, -5$$

$y' = 0$: $3x + 10 = 0 \Rightarrow \left(-\frac{10}{3}, \frac{5\sqrt[3]{4}}{3}\right)$ H.T.

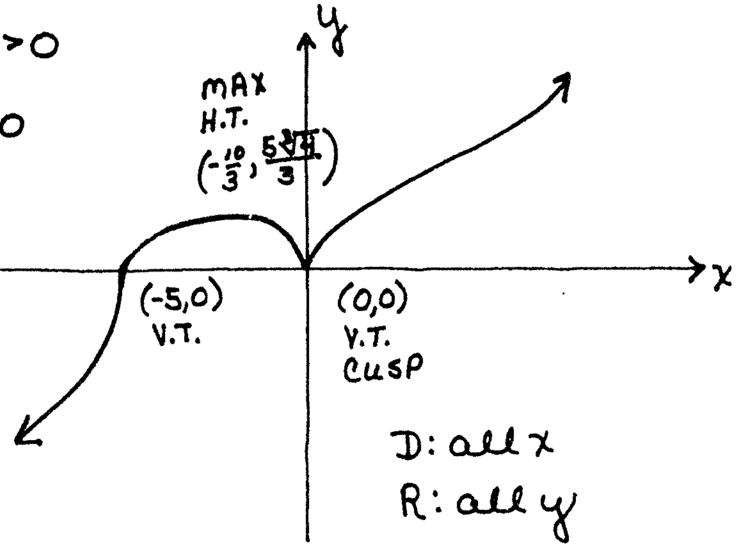
$$\left[\text{at } x = -\tfrac{10}{3}:\ y = \left(\tfrac{100}{9}\right)^{1/3}\left(-\tfrac{10}{3}+\tfrac{15}{3}\right)^{1/3} = \left(\tfrac{100}{9}\cdot\tfrac{5}{3}\right)^{1/3} = \left(\tfrac{4\cdot 125}{27}\right)^{1/3} = \tfrac{5}{3}\sqrt[3]{4}\right]$$

sign of y'



increasing: $x < -5$,
$-5 < x < -\frac{10}{3}$, $x > 0$

decreasing: $-\frac{10}{3} < x < 0$



D: all x
R: all y

3.) $y = x(3x+10)^{2/3}$

intercepts: $\underline{(0,0)}$ $\underline{(-\frac{10}{3},0)}$

$y' = \frac{5(x+2)}{(3x+10)^{1/3}} \Rightarrow x = -\frac{10}{3}$ is V.T. ; $x = -2$ is H.T.

$\underline{(-\frac{10}{3}, 0)}$ V.T. $\underline{(-2, -4\sqrt[3]{2})}$ H.T.

sign of y':

$+$ | $-$ | $+$

$-\frac{10}{3}$ MAX, -2 MIN

increasing: $x < -\frac{10}{3}$, $x > -2$

decreasing: $-\frac{10}{3} < x < -2$

$$y'' = \frac{(3x+10)^{1/3}(5) - 5(x+2)\left[\frac{1}{\cancel{3}}(3x+10)^{-2/3}(\cancel{3})\right]}{(3x+10)^{2/3}} = \frac{5(3x+10)-5(x+2)}{(3x+10)^{4/3}}$$

$y'' = \frac{10(x+4)}{(3x+10)^{4/3}} \Rightarrow x = -4$

sign of y'':

$-$ | $+$ | $+$

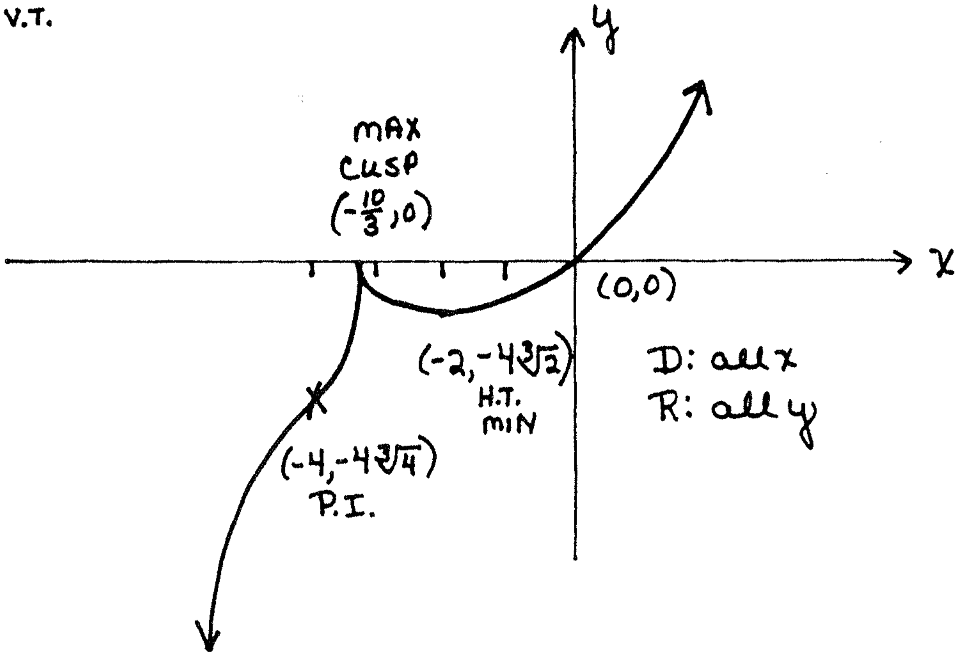
-4, $-\frac{10}{3}$ V.T.

$x = -4$ is P.I

$\underline{(-4, -4\sqrt[3]{4})}$

concave up: $-4 < x < -\frac{10}{3}$, $x > -\frac{10}{3}$

cancave down: $x < -4$



D: all x

R: all y

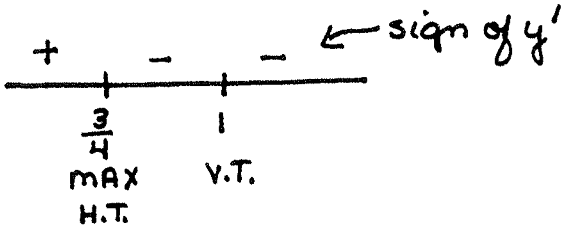
4.) $y = x(1-x)^{1/3}$

intercepts: $\underline{(0,0)}$ and $\underline{(1,0)}$

$$y' = x\left[\tfrac{1}{3}(1-x)^{-2/3}(-1)\right] + (1-x)^{1/3} = \frac{-x}{3(1-x)^{2/3}} + (1-x)^{1/3} = \frac{-4x+3}{3(1-x)^{2/3}}$$

$x = 1$ is V.T. $\quad x = \frac{3}{4}$ is H.T.

$\underline{(1,0)} \qquad \underline{\left(\frac{3}{4}, \frac{3\sqrt[3]{2}}{8}\right)}$

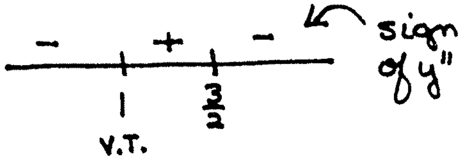


inc. $x < \frac{3}{4}$

dec. $\frac{3}{4} < x < 1, \ x > 1$

$$y'' = \frac{3(1-x)^{2/3}(-4) - (-4x+3)\left[2(1-x)^{-1/3}(-1)\right]}{9(1-x)^{4/3}} = \frac{-12(1-x)^{2/3} + \frac{2(-4x+3)}{(1-x)^{1/3}}}{9(1-x)^{4/3}}$$

$$y'' = \frac{-12(1-x) + 2(-4x+3)}{9(1-x)^{4/3}(1-x)^{1/3}} = \frac{4x-6}{9(1-x)^{5/3}} \qquad y'' = 0: \quad x = 3/2$$



$\Rightarrow \underline{\left(\frac{3}{2}, -\frac{3\sqrt[3]{4}}{4}\right)}$ is P.I.

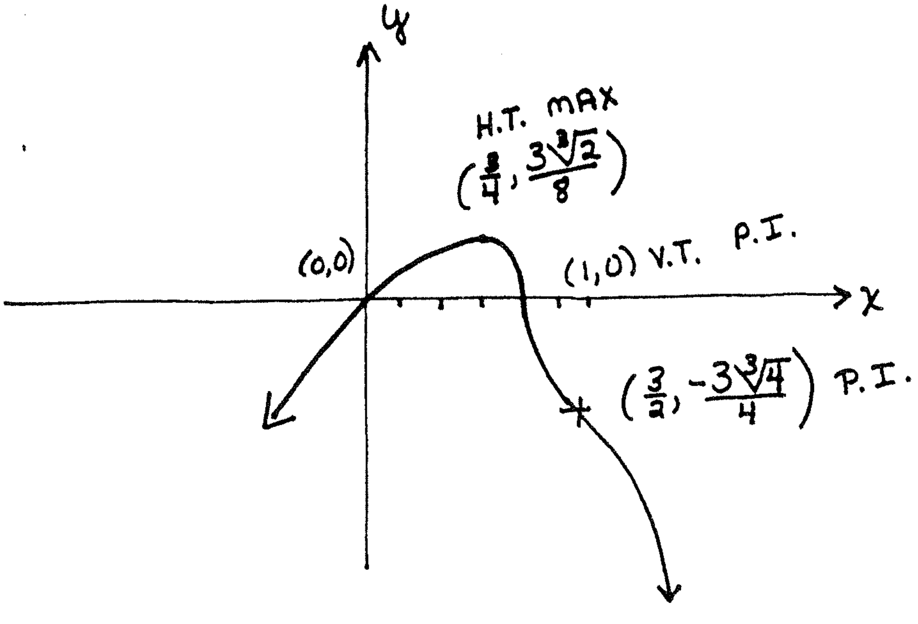
concave up: $1 < x < \frac{3}{2}$

concave down: $x < 1$

$x > \frac{3}{2}$

D: all x

R: $y \leq \frac{3\sqrt[3]{2}}{8}$



5.) $y = \frac{x^{2/3}}{x-8}$ [Do not check concavity.]

intercept: $\underline{(0,0)}$

asymptotes:

$\lim_{x \to 8^-} \frac{x^{2/3}}{x-8} = -\infty$ $\lim_{x \to 8^+} \frac{x^{2/3}}{x-8} = +\infty$ $\underline{x=8}$ is H.A.

$\lim_{x \to -\infty} \frac{\frac{1}{x^{1/3}}}{1-\frac{8}{x}} = 0$ $\lim_{x \to +\infty} \frac{\frac{1}{x^{1/3}}}{1-\frac{8}{x}} = 0$ $\underline{y=0}$ is V.A.

$$y' = \frac{(x-8)\left[\frac{2}{3}x^{-1/3}\right] - x^{2/3}}{(x-8)^2} = \frac{\frac{2(x-8)}{3x^{1/3}} - x^{2/3}}{(x-8)^2} = \frac{2(x-8)-3x}{3x^{1/3}(x-8)^2}$$

y' is undefined at $x=8$ (asymptote) and at $x=0$ (vertical tangent) $\underline{(0,0)}$ is V.T.

$y' = \frac{-(x+16)}{3x^{1/3}(x-8)^2} \Rightarrow x=-16$ is H.T. $\underline{\left(-16, \frac{-\sqrt[3]{4}}{6}\right)}$

sign of y':

− + − −

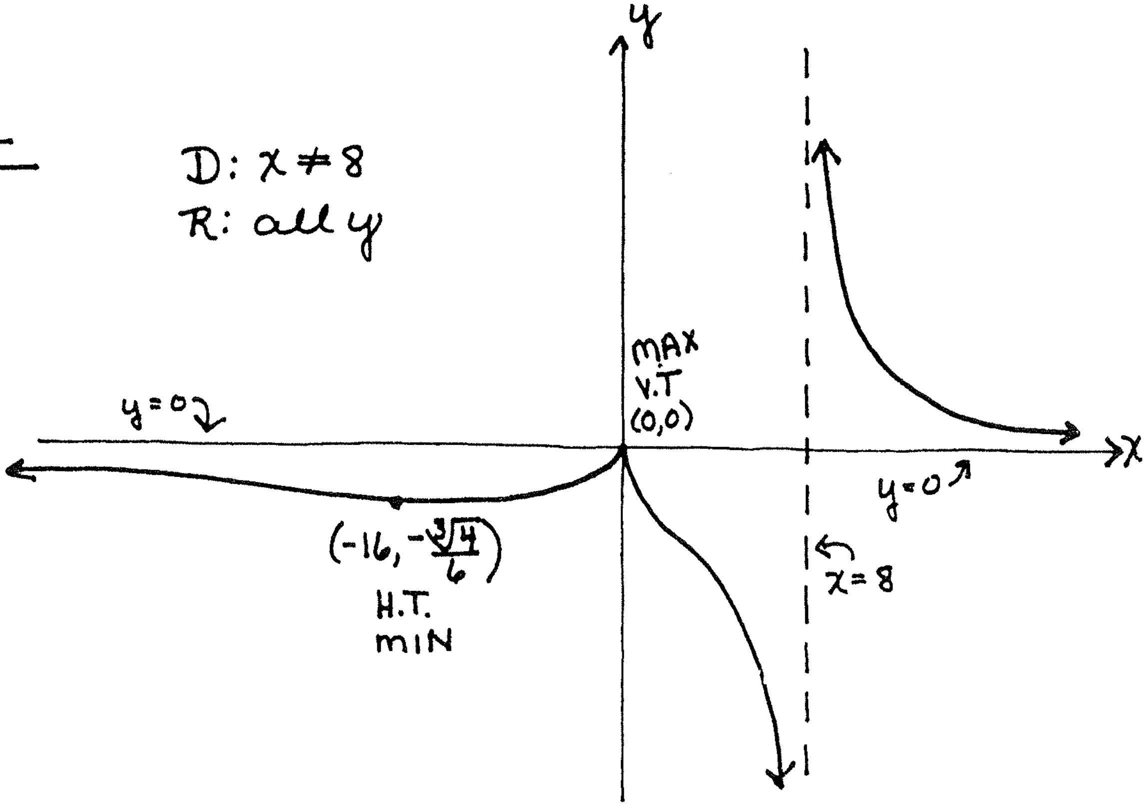
-16 0 8

H.T. V.T. V.A.

MIN MAX

D: $x \neq 8$

R: all y



Exercises Part IV

Sketch the graphs of the following. Include all important aspects.

1.) $y = (1+x)^{2/3}(x-4)$ 2.) $y = \frac{(x-1)^{2/3}}{x}$ (omit concavity)

3.) $y = \frac{x^{2/3}}{x-3}$ (omit concavity)

4.) $y = x^{2/3}(x-4)^2 - 4$ (do not check for x-intercepts or concavity)

5.) $y = x^{1/3}(4-x)^{2/3} + 2$ (do not check for x-intercepts or concavity)

6.) $y = x^3(9x+11)^{2/3}$ (omit concavity)

7.) $y = (x-2)^{2/3}(2x+1)$ (omit concavity)

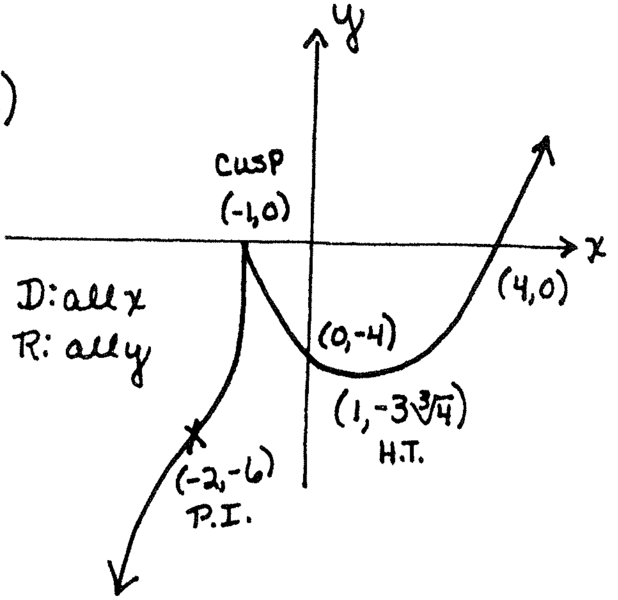
8.) $y = x^{1/3}(x+1)^{2/3}$ (omit concavity)

9.) $y = x^2(x+2)^{2/3}$ (omit concavity)

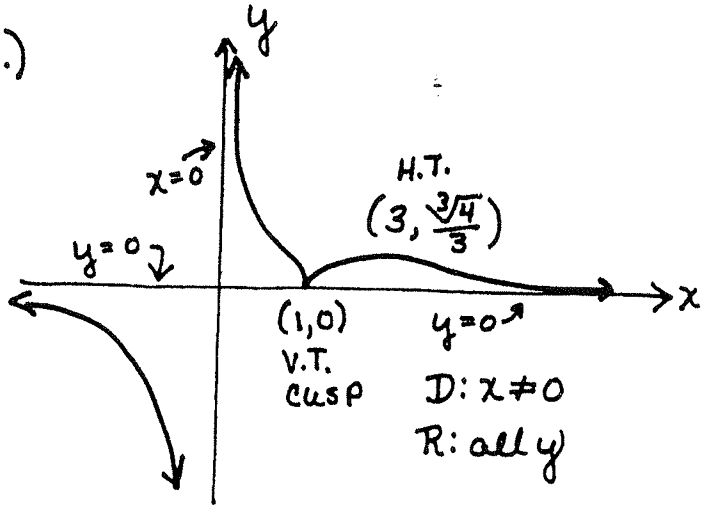
10.) $y = x^{1/3}(x+4)$

11.) $y = \frac{-x}{(x-1)^2}$ 12.) $y = x^3 - x^2 - x + 1$

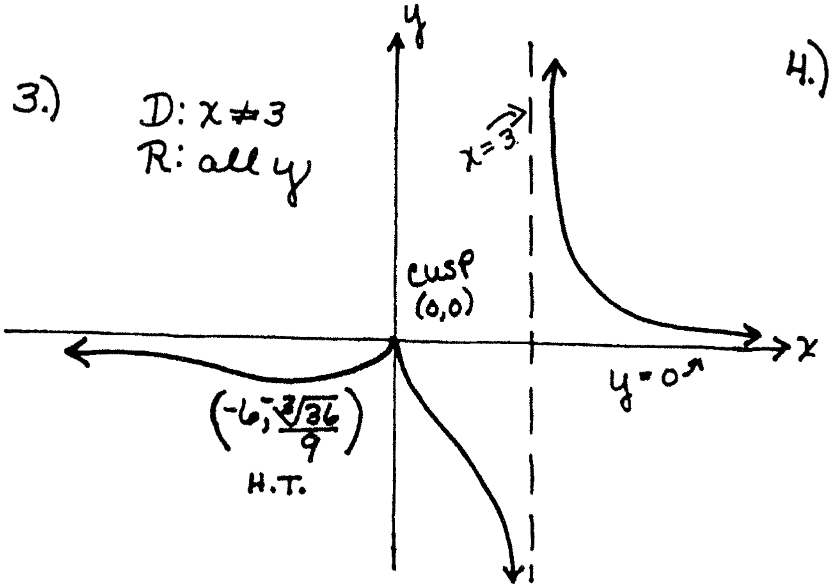
1.)


y
x
cusp
(-1,0)
D: all x
R: all y
(4,0)
(0,-4)
$(1,-3\sqrt[3]{4})$
H.T.
(-2,-6)
P.I.

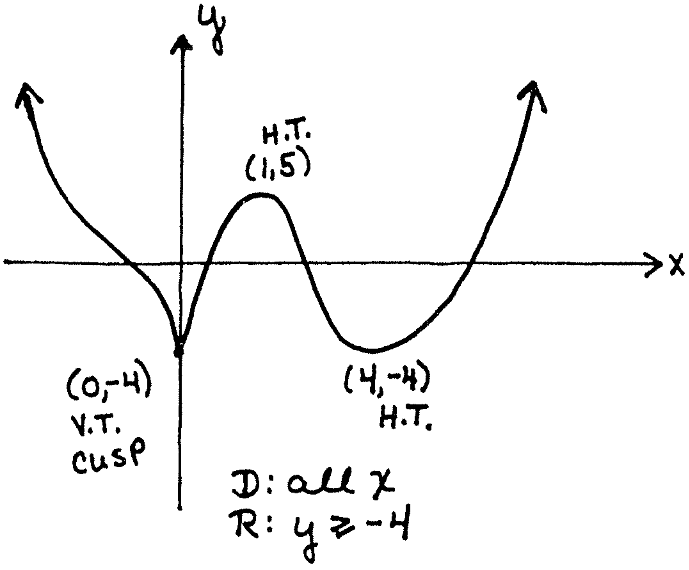
2.)


y
x
x=0
H.T.
$(3,\frac{\sqrt[3]{4}}{3})$
y=0
(1,0)
V.T.
cusp
y=0
D: x≠0
R: all y

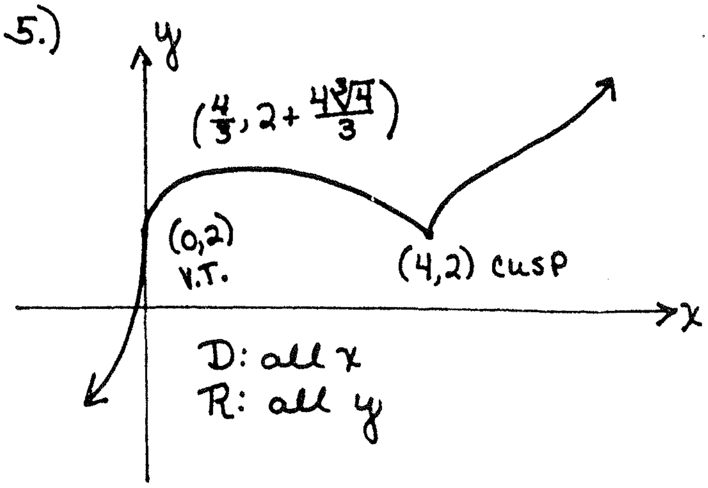
3.)


y
x
D: x≠3
R: all y
x=3
cusp
(0,0)
$(-6,-\frac{\sqrt[3]{36}}{9})$
H.T.
y=0

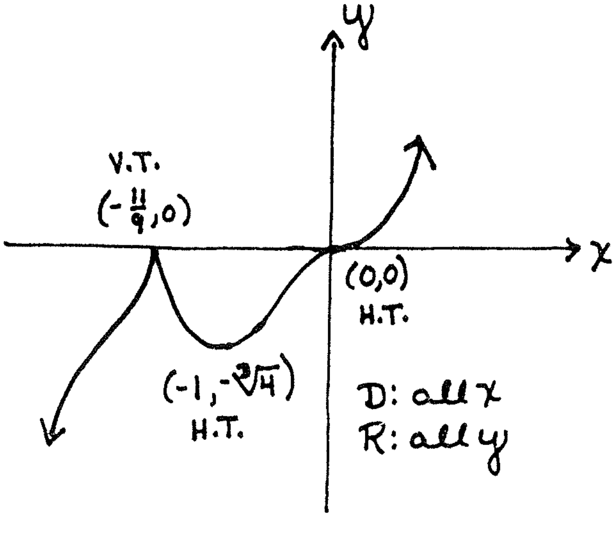
4.)


y
x
H.T.
(1,5)
(0,-4)
V.T.
cusp
(4,-4)
H.T.
D: all x
R: y≥-4

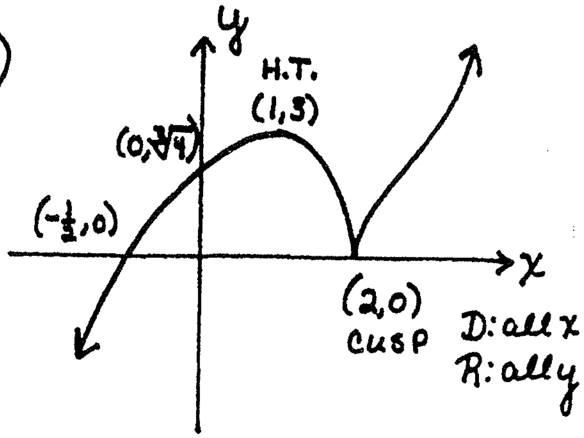
5.)


y
x
$(\frac{4}{3},2+\frac{4\sqrt[3]{4}}{3})$
(0,2)
V.T.
(4,2) cusp
D: all x
R: all y

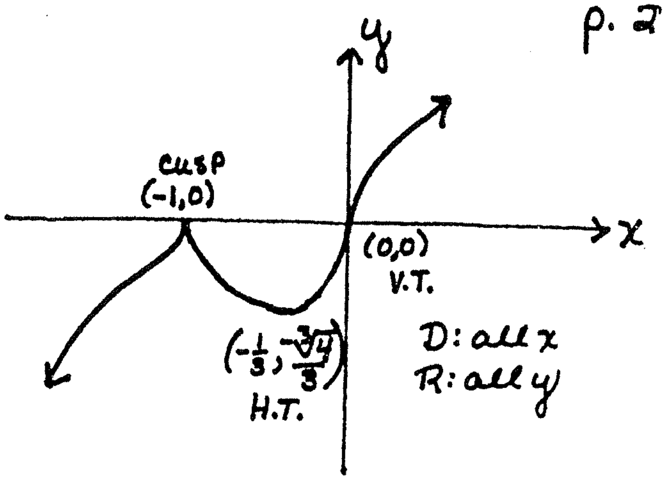
6.)


y
x
V.T.
$(-\frac{11}{9},0)$
(0,0)
H.T.
$(-1,-\sqrt[3]{4})$
H.T.
D: all x
R: all y

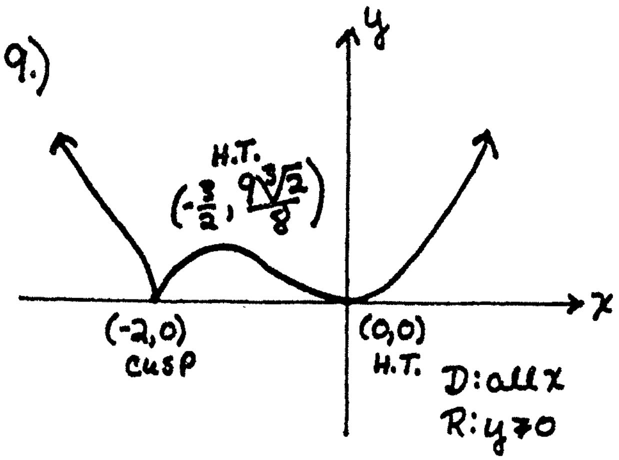
7.)


y
x
H.T.
(1,3)
(0,∛4)
(-1/2,0)
(2,0)
cusp
D: all x
R: all y

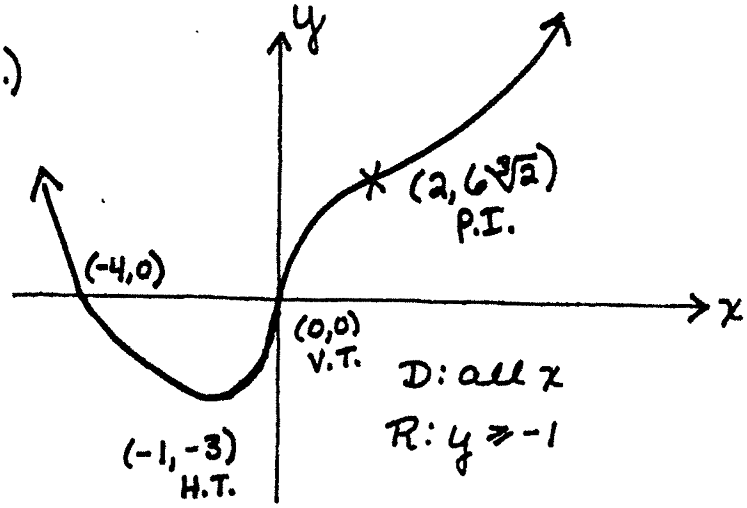
8.)


y
x
cusp
(-1,0)
(0,0)
V.T.
(-1/3, -∛4/3)
H.T.
D: all x
R: all y

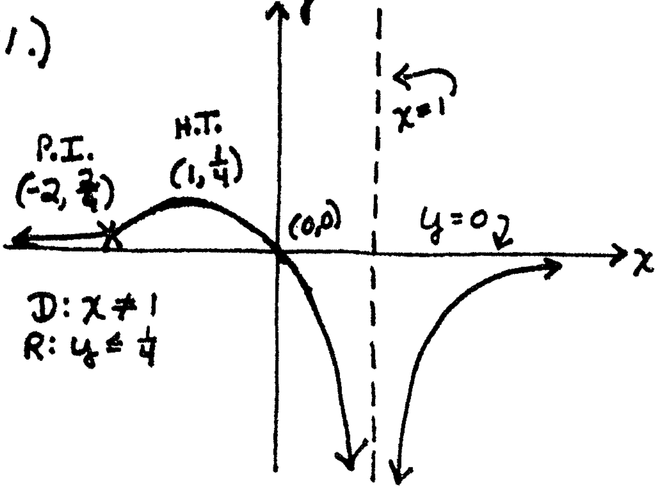
9.)


y
x
H.T.
(-3/2, 9∛2/8)
(-2,0)
cusp
(0,0)
H.T.
D: all x
R: y≥0

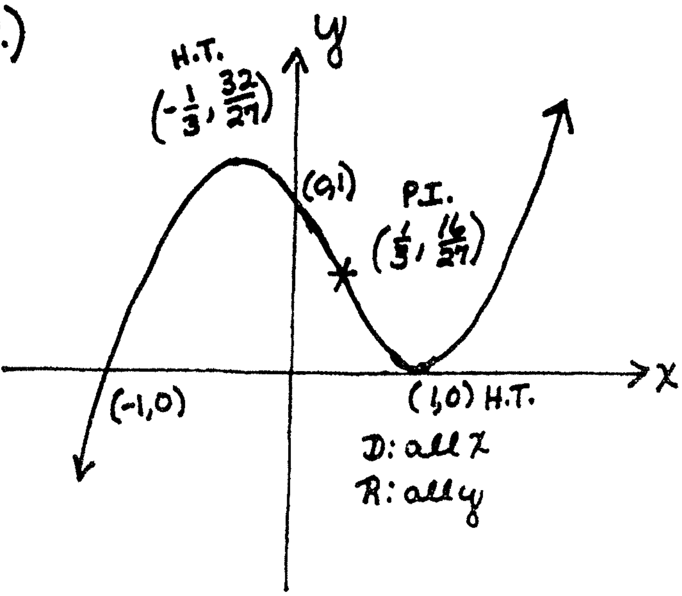
10.)


y
x
(2, 6∛2)
P.I.
(-4,0)
(0,0)
V.T.
(-1,-3)
H.T.
D: all x
R: y ≥ -1

11.)


y
x
x=1
P.I.
(-2, 7/4)
H.T.
(1, 1/4)
(0,0)
y=0
D: x ≠ 1
R: y ≤ 1/4

12.)


y
x
H.T.
(-1/3, 32/27)
(0,1)
P.I.
(1/3, 16/27)
(-1,0)
(1,0) H.T.
D: all x
R: all y