Computing Integrals: "The easy way"

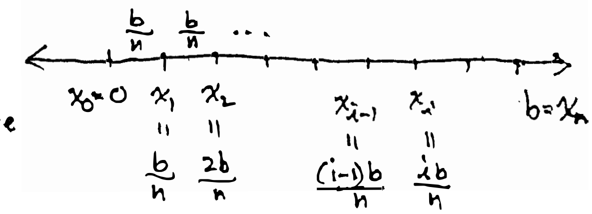
Under certain circumstances, one may compute a definite integral by evaluating only one limit of Riemann Sums.

Dividing the Interval (Partition)

Two ways turn out to be convenient.

I. Regular division of $[0, b]$ in Arithmetic Progression:

- n intervals;
- the difference between successive points is the same.



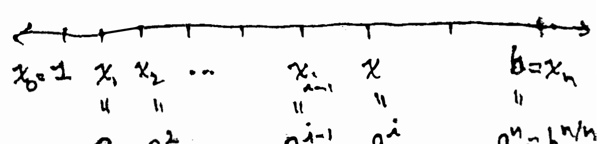
$$\boxed{x_i = \frac{ib}{n}}$$

N.B. $\Delta x_i \to 0$ as $n \to \infty$.

$$\boxed{\Delta x_i = \frac{b}{n}}$$

II. Division of $[1, b]$ in Geometric Progression:

- n intervals;
- the ratio between successive points is the same.



$$\boxed{x_i = b^{i/n}} = q^i, \text{ where } q = \sqrt[n]{b}.$$

$$\boxed{\Delta x_i = \left(b^{i/n} - b^{(i-1)/n}\right)} = q^i - q^{i-1}.$$

N.B. $\Delta x_i = b^{i/n} - b^{(i-1)/n} = b^{(i-1)/n}\left(\sqrt[n]{b} - 1\right) \to 0$ as $n \to \infty$.

1. $\int_0^b 3x^2\,dx$.

Recall formulas
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Regular partition: $x_i = \frac{ib}{n}$, $\Delta x_i = \frac{b}{n}$.

Riemann Sum: $\bar{x}_i = x_i = \frac{ib}{n}$; $f(x) = 3x^2$.

$$\sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i = \sum_{i=1}^{n} 3\left(\frac{ib}{n}\right)^2\left(\frac{b}{n}\right) = \sum_{i=1}^{n} 3\,\frac{i^2 b^3}{n^3}$$

$$= \frac{3b^3}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{3b^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{b^3}{2}\cdot\frac{(n+1)(2n+1)}{n^2}$$

Limit: $\lim \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i = \lim_{n\to\infty} \frac{b^3}{2}\cdot\frac{(n+1)(2n+1)}{n^2}$

$$= \frac{b^3}{2}\cdot 2 = b^3.$$

$\therefore \int_0^b 3x^2\,dx = b^3$.

Example: $\int_1^3 3x^2\,dx = \int_0^3 3x^2\,dx - \int_0^1 3x^2\,dx$

$$= 3^3 - 1^3 = 26.$$

Math 111 Integrals the easy way

Recall: Geometric Sum
$$\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r-1}$$

2. $\int_1^b x^p \, dx$, p = positive integer.

Geometric division: $x_i = b^{i/n} = q^i$, where $q = \sqrt[n]{b}$.

$$\Delta x_i = b^{i/n} - b^{(i-1)/n} = q^i - q^{i-1}.$$

Riemann Sum: $\bar{x}_i = x_{i-1} = q^{i-1}$; $f(x) = x^p$.

$k = i-1$; $i=1 \Rightarrow k=0$, $i=n \Rightarrow k=n-1$.

$$R.S. = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \sum_{i=1}^{n} (q^{i-1})^p (q^i - q^{i-1}) = \sum_{k=0}^{n-1} q^{kp}(q^{k+1} - q^k)$$

$$= \sum_{k=0}^{n-1} q^{kp} q^k (q-1) = (q-1)\sum_{k=0}^{n-1} q^{(p+1)k}$$

$$= (q-1)\cdot\frac{(q^{p+1})^n - 1}{q^{p+1}-1} \quad (\text{Geometric sum w/ } r = q^{p+1})$$

- First $(q^{p+1})^n = (q^n)^{p+1} = ((\sqrt[n]{b})^n)^{p+1} = b^{p+1}$;

second $q^{p+1} - 1 = (q-1)(1+q+q^2+\cdots+q^p)$ [$p+1$ terms in the sum].

Therefore ...

$$= \frac{\cancel{(q-1)}(b^{p+1}-1)}{\cancel{(q-1)}(1+q+q^2+\cdots+q^p)}$$

Limit: $\lim \sum f(\bar{x}_i)\Delta x_i = \lim_{n\to\infty} \frac{b^{p+1}-1}{(1+q+q^2+\cdots+q^p)}$ $(q = \sqrt[n]{b})$.

Since $\lim_{n\to\infty} q = \lim_{n\to\infty} b^{1/n} = b^0 = 1$, therefore ...

$$RS = \frac{b^{p+1}-1}{p+1}$$

$\therefore \int_1^b x^p\,dx = \frac{b^{p+1}-1}{p+1}$. $\therefore \int_a^b x^p\,dx = \frac{b^{p+1}}{p+1} - \frac{a^{p+1}}{p+1}$.

3. $\int_0^b e^x \, dx$.

Regular division: $x_i = \frac{ib}{n}$, $\Delta x_i = \frac{b}{n}$.

Riemann Sum: $\bar{x}_i = x_{i-1} = \frac{(i-1)b}{n}$; $f(x) = e^x$.

$k = i-1$; $i=1 \Rightarrow k=0$, $i=n \Rightarrow k=n-1$.

$$\sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i = \sum_{i=1}^{n} e^{(i-1)b/n} \cdot \frac{b}{n} = \sum_{k=0}^{n-1} e^{kb/n} \cdot \frac{b}{n}$$

$$= \frac{b}{n} \sum_{k=0}^{n-1} \left(e^{b/n}\right)^k = \frac{b}{n} \, \frac{\left(e^{b/n}\right)^n - 1}{e^{b/n} - 1} \quad \left(\text{geom. sum w/ } r = e^{b/n}\right)$$

$$= (e^b - 1) \cdot \frac{b/n}{e^{b/n} - 1}$$

Limit: $\lim \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i = \lim_{n \to \infty} \left[(e^b - 1) \cdot \frac{b/n}{e^{b/n} - 1} \right]$

$$= (e^b - 1) \left[\lim_{n \to \infty} \frac{e^{b/n} - 1}{b/n} \right]^{-1} \quad \left(\begin{array}{c} h = b/n \\ h \to 0 \text{ as } n \to \infty \end{array} \right)$$

$$= (e^b - 1) \left[\lim_{h \to 0} \frac{e^h - 1}{h} \right]^{-1}$$

$$= (e^b - 1) \cdot [1]^{-1} = e^b - 1 .$$

$\therefore \int_0^b e^x \, dx = e^b - 1$.

Example: $\int_a^b e^x \, dx = \int_0^b e^x \, dx - \int_0^a e^x \, dx = e^b - e^a$.

4. $\int_0^b \sin x \, dx$.

Recall
- $2\sin A \sin B = \cos(A-B) - \cos(A+B)$
- telescoping sums

Regular division: $x_i = \frac{ib}{n}$, $\Delta x_i = \frac{b}{n}$.

Riemann Sum: $\bar{x}_i = x_i = \frac{ib}{n}$; $f(x) = \sin x$.

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \sum_{i=1}^{n} \sin\left(\frac{ib}{n}\right)\frac{b}{n} = \sum_{i=1}^{n} \sin(ih)\cdot h \quad (h = b/n)$$

Key trick: multiply by $\frac{2\sin\left(\frac{h}{2}\right)}{2\sin\left(\frac{h}{2}\right)}$. Thus...

$$= \sum_{i=1}^{n} \frac{h\left[2\sin(ih)\sin\left(\frac{h}{2}\right)\right]}{2\sin\left(\frac{h}{2}\right)}$$

$$= \frac{h}{2\sin\left(\frac{h}{2}\right)} \sum_{i=1}^{n} \left\{\cos\left(\left(i-\tfrac{1}{2}\right)h\right) - \cos\left(\left(i+\tfrac{1}{2}\right)h\right)\right\} \quad \left(\text{by trig. identity}\right)$$

$$= \frac{h}{2\sin\left(\frac{h}{2}\right)} \left\{\cos\left(\tfrac{1}{2}h\right) - \cancel{\cos\left(\tfrac{3}{2}h\right)} + \cancel{\cos\left(\tfrac{3}{2}h\right)} - \cancel{\cos\left(\tfrac{5}{2}h\right)} + \cdots + \cancel{\cos\left(\tfrac{2n-1}{2}\cdot h\right)} - \cos\left(\tfrac{2n+1}{2}\cdot h\right)\right\}$$

$$= \frac{h}{2\sin\left(\frac{h}{2}\right)} \left\{\cos\left(\tfrac{1}{2}h\right) - \cos\left(nh + \tfrac{1}{2}h\right)\right\}$$

$$= \frac{h}{2\sin\left(\frac{h}{2}\right)} \left\{\cos\left(\tfrac{1}{2}h\right) - \cos\left(b + \tfrac{1}{2}h\right)\right\} \quad (nh = b)$$

Limit: $\lim\limits_{h\to 0} \frac{h}{2\sin\left(\frac{h}{2}\right)} = 1$ (and $h \to 0$ as $n \to \infty$).

$$\therefore \lim \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \lim_{h\to 0} \frac{h}{2\sin\frac{h}{2}} \left\{\cos\left(\tfrac{1}{2}h\right) - \cos\left(b + \tfrac{1}{2}h\right)\right\}$$

$$= 1\cdot\{\cos 0 - \cos b\} = 1 - \cos b.$$

Example: $\int_a^b \sin x \, dx = -\cos b - (-\cos a)$.

5. $\int_1^b \frac{1}{x}\,dx$

Geometric division: $x_i = b^{i/n} = q^i$, where $q = \sqrt[n]{b}$.

$$\Delta x_i = b^{i/n} - b^{(i-1)/n} = q^i - q^{i-1}.$$

Riemann Sum: $\bar{x}_i = x_i = q^i$; $f(x) = \frac{1}{x}$.

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \sum_{i=1}^{n} \frac{1}{q^i}\left(q^i - q^{i-1}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{q-1}{q}\right)$$

$$= n\left(\frac{q-1}{q}\right) = n\left(\frac{\sqrt[n]{b}-1}{\sqrt[n]{b}}\right)$$

Limit: $\lim \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \lim_{n\to\infty} n\left(\frac{b^{1/n}-1}{b^{1/n}}\right)$

$$= \lim_{n\to\infty} b^{-1/n} \cdot \lim_{n\to\infty} n\left(b^{1/n}-1\right)$$

Now $\lim_{n\to\infty} b^{-1/n} = b^0 = 1$.

Also $b^{1/n} = e^{\frac{1}{n}\ln b}$; so let $h = \frac{1}{n}\ln b$. Then $n = \frac{1}{h}\ln b$ and $b^{1/n} = e^h$.

As $n\to\infty$, $h\to 0$. Therefore $\lim_{n\to\infty} n\left(b^{1/n}-1\right) = \lim_{h\to 0} \frac{1}{h}\ln b\left(e^h - 1\right) =$

$\ln b \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = \ln b$.

Therefore.. $= 1 \cdot \ln b = \ln b$.

$\therefore \int_1^b \frac{1}{x}\,dx = \ln b$.

Example: $\int_a^b \frac{1}{x}\,dx = \ln b - \ln a$.

6. $\int_1^b \frac{1}{\sqrt{x}}\,dx$.

Geometric division: $x_i = b^{i/n} = q^i$, where $q = \sqrt[n]{b}$.

$\Delta x_i = b^{i/n} - b^{(i-1)/n} = q^i - q^{i-1}$.

Riemann Sum: $\bar{x}_i = x_{i-1} = q^{i-1}$; $f(x) = \frac{1}{\sqrt{x}}$.

$k = i-1$; $i=1 \Rightarrow k=0$, $i=n \Rightarrow k=n-1$.

$$\sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i = \sum_{i=1}^{n} (q^{i-1})^{-\frac{1}{2}}(q^i - q^{i-1}) = \sum_{k=0}^{n-1} q^{-k/2}(q^{k+1}-q^k)$$

$$= \sum_{k=0}^{n-1} q^{-k/2} q^k (q-1)$$

$$= (q-1)\sum_{k=0}^{n-1} q^{k/2} = (q-1)\sum_{k=0}^{n-1} (q^{1/2})^k$$

$$= (q-1)\cdot\frac{(q^{1/2})^n - 1}{q^{1/2}-1}\cdot\frac{q^{1/2}+1}{q^{1/2}+1} = \frac{\cancel{(q-1)}\,((q^n)^{1/2}-1)(q^{1/2}+1)}{\cancel{(q-1)}}$$

$$= (b^{1/2}-1)(b^{1/2n}+1).$$

Limit: $$\lim \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i = \lim_{n\to\infty}\left[(b^{1/2}-1)(b^{1/2n}+1)\right]$$

$$= (b^{1/2}-1)\lim_{n\to\infty}(b^{\frac{1}{2n}}+1)$$

$$= (b^{1/2}-1)\cdot(1+1)$$

$$= 2b^{1/2}-2.$$

$$\therefore \int_1^b \frac{1}{\sqrt{x}}\,dx = 2b^{1/2}-2.$$

Example: $$\int_a^b \frac{1}{\sqrt{x}}\,dx = 2\sqrt{b} - 2\sqrt{a}$$

$$\int_4^{25} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{25} - 2\sqrt{4} = 6.$$

7. $\int_0^b \cos 2x \, dx$.

Recall:
- $2\cos A \sin B = -[\sin(A-B) - \sin(A+B)]$
- telescoping sums

Regular division: $x_i = \frac{ib}{n} = ih$, where $h = \frac{b}{n}$.

$$\Delta x_i = \frac{b}{n} = h.$$

Riemann Sum: $\bar{x}_i = x_i \overset{=ih}{} ; f(x) = \cos 2x$

$$\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \sum_{i=1}^{n} \cos(2ih)\cdot h$$

Key trick: multiply by $\frac{2\sin(h)}{2\sin(h)}$. (If $\cos kx$, use $\frac{2\sin\left(\frac{kh}{2}\right)}{2\sin\left(\frac{kh}{2}\right)}$.)

$$= \sum_{i=1}^{n} \frac{h[2\cos(2ih)\sin(h)]}{2\sin(h)}$$

$$= \frac{h}{2\sin(h)} \sum_{i=1}^{n} -\left\{\sin((2i-1)h) - \sin((2i+1)h)\right\}$$

$$= \frac{-h}{2\sin h}\cdot\left\{\sin(h) - \cancel{\sin(3h)} + \cancel{\sin(3h)} - \cancel{\sin(5h)} + \cdots + \cancel{\sin((2n-1)h)} - \sin((2n+1)h)\right\}$$

$$= -\frac{h}{2\sin h}\left\{\sin(h) - \sin(2nh + h)\right\}$$

$$= -\frac{h}{2\sin h}\left\{\sin h - \sin(2b+h)\right\} \quad \text{since } 2nh = 2b$$

Limit: $\lim \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \lim_{h\to 0} \frac{h}{2\sin h}\left\{\sin(2b+h) - \sin(h)\right\}$ (since as $n\to\infty$, $h\to 0$)

$$= \frac{1}{2}\lim_{h\to 0}\frac{h}{\sin h}\lim_{h\to 0}\left\{\sin(2b+h) - \sin(h)\right\}$$

$$= \frac{1}{2}\cdot 1\cdot\left\{\sin 2b - \sin 0\right\}.$$

$$\therefore \int_0^b \cos 2x\,dx = \frac{1}{2}\sin 2b.$$

Example: $\int_a^b \cos 2x\,dx = \frac{1}{2}\sin 2b - \frac{1}{2}\sin 2a$.